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INCOME-SHARE ELASTICITY AND THE SIZE DISTRIBUTION OF INCOME*

BY J. ESTEBAN**

1. INTRODUCTION

The main purpose of this paper is to introduce the notion of Income-Share Elasticity as a convenient description of the size distribution of income. This concept, as an alternative to density functions, offers many advantages from both the theoretical and the applied point of view. On the one hand, it can help to capture some common features in seemingly unrelated density functions. On the other hand, it leads to a rather different approach to the description of the size distribution of income.

Empirical research on the size distribution of income has been mainly confined to the descriptive task of fitting density functions to data. Most contributions consist of introducing a new density function and showing that it fits better than the functions previously considered. But there seems to be no clear rationale with regard to the search for new functions. This situation is essentially due to the lack of a theory on the size distribution of income, which is bound to persist unless we change the way of analysing empirical data. We must admit that the knowledge that, for instance, the Gamma function gives a better fit than the Log-Normal is not very inspiring. An alternative approach could be that of the identification of stylized facts about the distribution of individual incomes. Indeed, if the size distribution is to be analyzed, the data must show some sort of stable structure. Therefore, it would seem natural to identify these regularities for applied work. Once these stylized facts have been well established, we shall have a smaller set of density functions from which to choose.

As an illustration of this approach, we consider a simple case in which we hypothesize three stylized facts and show that the three-parameter Generalised Gamma is the only density function that behaves accordingly. These three hypotheses are: a) distributions satisfy a modified version of Mandelbrot's [1960] Weak Pareto Law, which we call the Weak-WPL; b) they have a mode; and c) the Income-Share Elasticities have a constant rate of decline. These hypotheses can be subject to direct, independent tests. However, it is not our purpose to claim that they do satisfactorily correspond to observed facts. We cannot expect to have determined the class of functions that fit actual income distributions. The aim of the paper is to introduce an alternative way of approaching the description of the size distribution of incomes.

We begin by defining the Income-Share Elasticity and showing that there is a

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one to one correspondence between these elasticities and density functions. In Section 3 we introduce the following three testable hypotheses: distributions satisfy the Weak-Weak Pareto Law, they have a mode and the Income-Share Elasticity has a constant rate of decline. We analyse the implications of this later hypothesis and find that a good number of the most commonly used distributions have Income-Share Elasticities with a constant rate of decline. Moreover, we find in Section 4 that the Generalized Gamma is fully characterized by these three hypothesized facts[1].

2. THE CHARACTERIZATION OF DENSITY FUNCTIONS BY MEANS OF THEIR INCOME-SHARE ELASTICITY

Let $\boldsymbol{F}$ be the set of density functions f with support $[a, b]$, $0 \leq a < b \leq +\infty$, such that: i) f has a finite mean, ii) f is C^1 in (a, b), and iii) $f(x) > 0$ for all $x \in (a, b)$, and $f(x) = 0$ otherwise.

First of all we will define income-share elasticity.

DEFINITION. (*Income-Share Elasticity*) *Let $\gamma(x, x+h)$ be the share of total income earned by individuals with incomes in the interval $[x, x+h]$. The Income-Share Elasticity at x, $\pi(x)$, of a given distribution is the limit when $h \to 0$ of the proportional change of γ with respect to x, that is*

$$\pi(x) = \operatorname*{Lim}_{h \to 0} \frac{d \log \gamma}{d \log x}.$$

Let $\pi\colon (a, b) \to \boldsymbol{R}$ be an income share elasticity and let us denote by $\Lambda\colon (a, b) \to \boldsymbol{R}$ the primitive of $[1-\pi(x)]/x$ in (a, b), that is

$$\Lambda(x) = \int [1-\pi(x)]/x dx + C.$$

Let us denote by $\boldsymbol{\Pi}(a, b)$ the set of admissible income-share elasticities, i.e. those satisfying: i) π is continuous on (a, b); ii) $\int_a^b \exp\{-\Lambda(x)\}\, dx < +\infty$ and iii) $\int_a^b x \exp\{-\Lambda(x)\} dx < +\infty$, where the integrals in ii) and iii) are improper integrals. Observe that if $b < \infty$, then (ii)$\Rightarrow$(iii). When $a \geq 1$, (iii)$\Rightarrow$(ii), and when $a < 1$ this implication does not hold true.

We shall now show that there is a one to one correspondence between income-share elasticities and density functions in $\boldsymbol{F}$.

PROPOSITION. *There is a one-to-one correspondence between $f \in \boldsymbol{F}(a, b)$ and $\pi \in \boldsymbol{\Pi}(a, b)$.*

[1] The Generalized Gamma was obtained by Stacy [1962] and by Amoroso [1924–1925] and D'Addario [1974] and recently fitted to income distributions by Kloek and van Dijk [1978] and McDonald [1984]. Unfortunately, none of the last two works can be used as an indirect test for the validity of our hypotheses.

PROOF. Consider a density function $f \in \boldsymbol{F}(a, b)$. The share of the total income earned by individuals with incomes in the income interval $[x, x+h]$ is

$$\gamma(x, x+h) = (1/\mu)\int_{x}^{x+h} yf(y)dy, \text{ where}$$

$$\mu = \int_{0}^{\infty} xf(x)dx \text{ is the mean income.}$$

The proportional change of γ with respect to x is given by

$$\frac{d \log \gamma\,(x, x+h)}{d \log x} = \frac{x(x+h)f(x+h) - x^2 f(x)}{\mu\gamma(x, x+h)}.$$

Letting h tend to zero, we finally obtain by L'Hopital's rule

$$\Pi(x) = \operatorname*{Lim}_{h\to 0} \frac{d \log \gamma}{d \log x} = 1 + \frac{xf'(x)}{f(x)}. \tag{1}$$

It is obvious that for any $f \in \boldsymbol{F}$, the corresponding π is continuous. Moreover,

$$\frac{1-\pi(x)}{x} = \frac{f'(x)}{f(x)}$$

and therefore

$$\Lambda(x) = \log f(x) + C.$$

It is thus immediate that ii) and iii) are both satisfied, so that $\pi \in \boldsymbol{\Pi}(a, b)$.

Consider now $\pi \in \boldsymbol{\Pi}(a, b)$ and define

$$f(x) = \begin{cases} \dfrac{\exp\{-\Lambda(x)\}}{\int_a^b \exp\{-\Lambda(x)\}dx} & \text{for } x \in (a, b) \\ 0 & \text{otherwise.} \end{cases}$$

It is immediate that $f \in \boldsymbol{F}(a, b)$ and that does not depend on the chosen Λ (i.e. on C).

Indeed this is a one-to-one correspondence. Suppose that $f,\ g \in \boldsymbol{F}(a, b)$ and $\pi_f = \pi_g$. Then we have that

$$\frac{f'(x)}{f(x)} = \frac{g'(x)}{g(x)} \qquad \text{for all} \quad x \in (a, b).$$

Thus it must be that $f(x) = Cg(x)$. But since both f and g are density functions we have that $f = g$.

Moreover, let $\pi,\ \pi^* \in \boldsymbol{\Pi}(a, b)$, with primitives Λ and Λ^*, respectively, and $f_\pi = f_{\pi^*}$. Then

$$\frac{\exp\{-\Lambda(x)\}}{\int_a^b \exp\{-\Lambda(x)\}dx} = \frac{\exp\{-\Lambda^*(x)\}}{\int_a^b \exp\{-\Lambda^*(x)\}dx} \text{ for all } x \in (a, b).$$

Thus

$$\exp\{-\Lambda(x)+\Lambda^*(x)\} = \frac{\int_a^b \exp\{-\Lambda(z)\}\,dz}{\int_a^b \exp\{-\Lambda^*(z)\}\,dz}.$$

And from this it follows that $\Lambda^*(x)-\Lambda(x)=C$. This completes the proof.

The income-share elasticity indicates the rate of change of total income at each income level. It seems useful, especially in applied problems, to think of density functions in terms of their corresponding π function. The Pareto, Gamma, and Normal density functions correspond to constant, lineal, and quadratic elasticities, respectively.

3. THREE HYPOTHESES

3.1. *The Weak-Weak Pareto Law.* Pareto was responsible for the first attempt at defining a general law that tried to explain the regularities of observed distributions. Let $1-F(x)$ be the percentage of individuals with incomes greater or equal to x. Then, the (strong) Pareto Law asserts that

$$1 - F(x) = \begin{cases} (x/x_0)^{-\alpha} & \text{when} \quad x_0 \leq x \leq \infty \\ 1 & \text{when} \quad x < x_0, \end{cases}$$

for some $\alpha>0$ and $x_0>0$ and the support of F is $[x_0, +\infty)$.

Available empirical evidence leaves little doubt that Pareto's Law, as it stands, does not account satisfactorily for a wide range of incomes. However, if our attention is restricted to the upper tail of the distributions, the evidence does not contradict the (Strong) Pareto Law, provided that the chosen x_0 is large enough. This suggests that actual distributions asymptotically approach (rather than coincide with) the Pareto distribution. In view of this fact, Mandelbrot [1960] proposed the Weak (asymptotic) Pareto Law (WPL). The WPL simply states that

$$\text{(3)} \qquad \lim_{x\to\infty} \frac{1-F(x)}{(x/x_0)^{-\alpha}} = 1,$$

for some $\alpha>0$ and $x_0>0$ and for F with support $[a, +\infty)$ and $a\geq 0$.

One way of characterising Pareto distributions is by the fact that total income at each income level falls at a constant rate. Therefore it seems natural to rephrase Mandelbrot's version of the WPL as

$$\text{(4)} \qquad \lim_{x\to\infty} \pi(x) = -\alpha$$

where π is as defined in (1).

Mandelbrot mentions in passing this alternative formulation and takes it as equivalent to the WPL[2]. But (4) is in fact weaker than (3), that is, (4) does not imply (3). Consider, for instance, the case $\pi(x) = -\alpha + 1/\log x$. Substituting into (2) it is a matter of simple calculation to check that while (4) is satisfied, (3) is not. Condition (4) therefore is not equivalent to (3) and we shall call it the Weak-Weak Pareto Law (WWPL).

Now we will introduce the assumption that income distributions follow the WWPL.

HYPOTHESIS 1. *Income distributions in* $\boldsymbol{F}(a, +\infty)$ *follow the WWPL, that is,*

$$\lim_{x \to \infty} \pi(x) = -\alpha, \quad \alpha > 0.$$

In contrast, an examination of the income-share elasticities corresponding to the density functions in current use in the analysis of income distributions shows that most of them do not obey the WWPL (this is shown in Section 3.3.). In other words, these distributions imply that the rate of change of total income at each income level falls too sharply at the upper tail.

3.2. *The Mode.* In view of the available evidence it seems quite natural to assume the existence of a mode in the interior of the interval of definition.

HYPOTHESIS 2. *Density functions* f *in* $\boldsymbol{F}(a, b)$ *have at least one mode in the interior of its support, that is* $f'(m) = 0$ *for some* $m \in (a, b)$.

It follows from Hypothesis 2 and equation (1) that $\pi(x) = 1$ has at least one solution in the interval (a, b).

3.3. *The Constant Rate of Decline of the Income-Share Elasticity.* We put forward the hypothesis that $\pi'(x)$ follows a constant proportional fall throughout the whole range of definition.

HYPOTHESIS 3. *Income-share elasticities have a constant rate of decline, i.e. either* $\pi'(x) = 0$ *or*

$$\frac{d \log \pi'(x)}{d \log x} = -(1+\varepsilon), \quad 0 < x < +\infty$$

for some $\varepsilon > -1$.

Upon integration we find that the family of income-share elasticities satisfying Hypothesis 3 is given by

$$(5) \qquad \begin{aligned} \pi(x) &= -\alpha + \beta/\log x, \quad \text{when} \quad \varepsilon = 0, \\ \pi(x) &= -\alpha + [\beta/|\varepsilon|]x^{-\varepsilon}, \quad \text{otherwise}, \end{aligned}$$

where α and β are constants of integration.

It is interesting to note that many well-known density functions have income-

[2] Mirrlees [1971] also takes (4) as equivalent to (3).

share elasticities which follow a constant rate of decline: Pareto ($\beta=0$), Exponential ($\alpha=-1$, $\beta=1$, $\varepsilon=-1$), Chi ($\alpha=-\nu$, $\beta=2$, $\varepsilon=-2$), Chi-square ($\alpha=-n/2$, $\beta=1/2$, $\varepsilon=-1$), Weibull ($\alpha=\varepsilon$, $\varepsilon<0$, $\beta=\varepsilon^2 a^\varepsilon$), Normal ($\alpha=-1$, $\beta=2$, $\varepsilon=-2$), Gamma ($\alpha<0$, $\varepsilon=-1$) and Lognormal ($\alpha=1/\varepsilon\sigma^2$, $\beta=\mu^\varepsilon/\sigma^2$, $\varepsilon\to 0$), among many others.

4. THE GENERALIZED GAMMA FUNCTION

Combining Hypotheses 1 and 3 one easily finds that both a and e must be positive, i.e. $\alpha>0$ and $\varepsilon>0$. If we now add Hypothesis 2 we find that there is a unique m that solves $\pi(m)=1$. This value of m is

$$m = [\beta/(1+\alpha)\varepsilon]^{1/\varepsilon}.$$

Given that $\alpha>0$ and $\varepsilon>0$, it follows be that $\beta>0$. Reparametrizing we finally have

$$\pi(x) = -\alpha + (1+\alpha)[x/m]^{-\varepsilon},\ \alpha > 1,\ \varepsilon > 0,\ m > 0. \tag{6}$$

This defines the family of income-share elasticities that satisfies Hypotheses 1, 2 and 3. Substituting into (2) we can obtain the corresponding three-parameter density function. This density function is

$$f(x) = [\varepsilon/m\Gamma(\alpha/\varepsilon)]\,[(1+\alpha)/\varepsilon]^{\alpha/\varepsilon}[x/m]^{-(1+\alpha)}\exp\{-[(1+\alpha)/\varepsilon][x/m]^{-\varepsilon}\} \tag{7}$$

where Γ is the Gamma function. The r-th moment is given by

$$E(x^r) = \frac{\Gamma[(\alpha-r)/\varepsilon]}{\Gamma[\alpha/\varepsilon]}\,[(1+\alpha)/\varepsilon]^{r/\varepsilon}\,m^r. \tag{8}$$

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REFERENCES

Amoroso, L., "Ricerche intorno alla Curva dei Rediti," *Matematica Pura ed Applicata*, series IV, 2, (1924–1925).

D'Addario, R., "Intorno ad una Funzione di Distribuzione," *Giornale degli Economisti e Annali di Economia*, 33, new series (1974), 205–214.

Kloek, T. and H. K. van Dijk, "Efficient Estimation of Income Distribution Parameters," *Journal of Econometrics*, 8 (1978), 61–74.

McDonald, J. B., "Some Generalized Functions for the Size Distribution of income," *Econometrica*, 52 (1984), 647–663.

Mandelbrot, B., "The Pareto-Lévy Law and the Distribution of Income," *International Economic Review*, 1 (1960), 79–106.

Mirrlees, J. A., "An Exploration in the Theory of Optimum Income Taxation," *Review of Economic Studies*, 38 (1971), 175–208.

Stacy, E. W., "A Generalization of the Gamma Distribution," *Annals of Mathematical Statistics*, 33 (1962), 1187–1192.

[3] For a finite mean we need $\alpha>1$ and for a finite variance $\alpha>2$.