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Notes, Comments and Letters to the Editor

Core Equivalence in an Overlapping Generations Model*

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In this paper, we place the "short term core" of Aliprantis and Burkinshaw in the chart of core-like sets in overlapping generations models. For the limit of replica economis, it is shown that the short term core is equivalent to Chae's short run core and therefore to the set of competitive equilibrium allocations in the special case where each generation consists of only one type of agents who consume the same consumption vector, but that in general the short term core is not equivalent to the set of competitive equilibrium allocations. Another result of the paper is a sufficient condition for a competitive equilibrium allocation to be in the core. *Journal of Economic Literature* Classification Numbers: D50, D90, C70. © 1993 Academic Press, Inc.

1. INTRODUCTION

In an environment where people get together to trade and leave after all trades are consummated, the artifact of competitive equilibrium seems to be a good approximation to reality. In some markets there are auctioneers who set prices to equate demand and supply. Stock markets and fish markets are prominent examples. In other markets which do not have auctioneers centralizing the transactions, people usually find their way to a situation resembling a competitive equilibrium. The invisible auctioneer is in fact the haggling process in the market. The equivalence between the set of competitive equilibrium allocations and the core is a theory explaining this phenomenon.

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In a different environment where trades continue without a certain end while people come and go, it is unclear whether the haggling process would lead to a situation resembling a competitive equilibrium. Furthermore, it is questionable whether a competitive equilibrium is a good approximation to reality. We regard the literature on the core in overlapping generations models as a ground work in answering these questions: Hendricks *et al.* [6], Kovenock [7], Esteban [4], Chae [2], Chae and Esteban [3], Esteban and Millán [5], and Aliprantis and Burkinshaw [1]. It has been noticed that the core equivalence does not hold in an overlapping generations model. In fact this is obvious from Samuelson's pioneering work [8] introducing an overlapping generations model, for he shows that the society as a whole can improve upon a competitive equilibrium allocation. An interesting question then is what core-like set is equivalent to the set of competitive equilibrium allocations. A solution to this question has been given by Chae [2]. He shows that the set of competitive equilibrium allocations is equivalent to a short run core, which is the limit of the cores of finite economies. The spirit of this result is quite different from the core equivalence for a finite economy. The core equivalence in the finite case shows positively that the haggling process leads to a situation resembling a competitive equilibrium. Its analogue in the overlapping generations model shows what limitations on improving coalitions are necessary to make a competitive equilibrium a stable outcome. Roughly speaking, competitive equilibrium allocations are those which cannot be improved upon in a finite horizon economy if one restricts the behavior of people who live both within and beyond this horizon. An interesting point is that restricting improving coalitions to be finite is not sufficient. Chae and Esteban [3] show that the bounded core, which is the set of allocations which cannot be improved upon by a finite coalition (or a bounded coalition in an economy with a continuum of agents), is in general larger than the set of competitive equilibrium allocations.

In a recent article, Aliprantis and Burkinshaw [1] present a result on the equivalence between a short term core, which we will call the bounded perturbation core (BPC), and the set of competitive equilibrium allocations for an overlapping generations model. They also claim that the result is true for a general overlapping generations model. One purpose of the present paper is to place the BPC in the chart of core-like sets in overlapping generations models. It will be shown in Section 3 that the result of Aliprantis and Burkinshaw [1] is not general. Indeed, the result holds only for a special case where each generation consists of only one type of agents who consume the same consumption vector. The basic problem is that in an overlapping generations model with heterogeneous agents in each generation, there are intertemporal trades among the members of the same generation in a competitive equilibrium, that is, there are savers and dissavers. The equivalence they assert fail unless these savings are zero. For the general equivalence, one should look at the short run core of Chae [2]. In fact, it will be also shown in Section 3 that the "short term core" of Aliprantis and Burkinshaw is equivalent to Chae's short run core in the case where their result is valid.

Another result of the present paper is a sufficient condition for a competitive equilibrium to be in the core of an overlapping generations model strengthening the results of Chae [2, Theorem 4.1] and Esteban and Millán [5, Proposition 2]. The condition requires that the present value of the young generation's endowments does not stay away from zero as time goes on.

2. Core-like Sets

We will investigate a discrete version of the pure exchange, overlapping generations model of Chae [2]. In each period $t = 1, 2, ..., \infty$, there is a finite set M^t of commodities. Each household h in generation t, denoted G^t, lives only in periods t and $t+1$. Put $M = \bigcup_{t=1}^{\infty} M^t$ and $G = \bigcup_{t=1}^{\infty} G^t$. A household may consume nonnegative amounts of commodities available during its life span, i.e., $c_h = (c_h^t, c_h^{t+1}) \in \mathbb{R}(M_h)_+$, where $M_h = M^t \cup M^{t+1}$, if $h \in G^t$. An assignment of consumptions for generation t in periods t and $t+1$ will be denoted by $c_t^t = (c_h^t)_{h \in G^t}$ and $c_t^{t+1} = (c_h^{t+1})_{h \in G^t}$, respectively. Also, let $c_t = (c_t^t, c_t^{t+1})$ and $c^t = (c_{t-1}^t, c_t^t)$. The preferences of household $h \in G$ are represented by a utility function $u_h\colon \mathbb{R}(M_h) \to \mathbb{R}$ which is continuous, quasiconcave, and strictly increasing. A household $h \in G^t$ has endowments over its life span, i.e., $e_h \in \mathbb{R}(M^t \cup M^{t+1})_+$. Assume, for $h \in G^t$, that $u_h(e_h) > u_h(c_h)$ if $c_h^t \notin \mathbb{R}(M^t)_{++}$. In particular, $e_h^t \in \mathbb{R}(M^t)_{++}$ for any $h \in G^t$, and thus $\sum_{h \in G} e_h \in \mathbb{R}(M)_{++}$. An allocation c is a feasible assignment, viz. $\sum_{h \in G} c_h = \sum_{h \in G} e_h$.

Denote a price system by $p \in \mathbb{R}(M)$. A pair (p, c) is called a *competitive equilibrium* if it satisfies

Budget Constraint: $pc_h \leqslant pe_h$ for all $h \in G$,

Undominatedness: $px_h > pe_h$ if $u_h(x_h) > u_h(c_h)$ for any $h \in G$,

Market Clearing: $p^m(\sum_{h \in G} e_h^m - \sum_{h \in G} c_h^m) = 0$ for any $m \in M$.

A nonempty subset of G is called a coalition. For any coalition S and assignment c, let $F_S(c)$ denote the set of resources which can be distributed among households in S so that S is made better off, viz.

$$F_S(c) = \{r \in \mathbb{R}_+^M;\ \textstyle\sum_{h \in S} x_h = r \text{ for some assignment } x \text{ such that } u_h(x_h) \geqslant u_h(c_h) \text{ for all } h \in S \text{ and } u_h(x_h) > u_h(c_h) \text{ for some } h \in S\}.$$

Also let $F_S^*(c)$ be the set of resources which can be distributed among households in S so that S is made better off with a finite perturbation of consumptions, viz.

$$F_S^*(c) = \{r \in \mathbb{R}_+^M;\ \sum_{h \in S} x_h = r \text{ for some assignment } x \text{ such that } u_h(x_h) \geqslant u_h(c_h) \text{ for all } h \in S,\ u_h(x_h) > u_h(c_h) \text{ for some } h \in S, \text{ and } x_h = c_h \text{ for all but a finite number of } h \in S\}.$$

Obviously, one has $F_S^*(c) \subset F_S(c)$ in general.

A coalition V is said to be relevant to another coalition S if households in S want resources initially held by V, i.e., for any allocation c there exists some $a > 0$ such that $a \sum_{h \in V} e_h + \sum_{h \in S} c_h \in F_S(c)$. A coalition W is said to be irreducible if for any two coalitions S and V such that $S \cap V = \phi$ and $S \cup V = W$, S is relevant to V. We assume that $G^1 \cup \cdots \cup G^t$ is irreducible for any $t \in T$.

An allocation c belongs to the *core* if it cannot be improved upon by any coalition, i.e., $\sum_{h \in S} e_h \notin F_S(c)$ for any coalition S. An allocation c belongs to the *bounded core* (or finite core) if it cannot be improved upon by any finite coalition, i.e., $\sum_{h \in S} e_h \notin F_S(c)$ $(= F_S^*(c))$ for any finite coalition S. An allocation c belongs to the *bounded perturbation core* if it cannot be improved upon by any coalition with a finite perturbation of consumptions, i.e., $\sum_{h \in S} e_h \notin F_S^*(c)$ for any coalition S. (The bounded perturbation core is what Aliprantis and Burkinshaw [1] call the "short term core.") An allocation c is *Pareto optimal* if c cannot be improved upon by the coalition of all households, i.e., $\sum_{h \in G} e_h \notin F_G(c)$. An allocation c is *short run Pareto optimal* if it cannot be improved upon by the coalition of all households with a finite perturbation of consumptions, i.e., $\sum_{h \in G} e_h \notin F_G^*(c)$.

A *t-economy* with respect to c_t^{t+1} is a t period economy with commodities in $M^1 \cup \cdots \cup M^t$ and households in $G^1 \cup \cdots \cup G^t$ where the characteristics of households in G^t are modified as follows:

(i) the consumption set of a household $h \in G^t$ is $\mathbb{R}(M^t)_+$,

(ii) household h prefers x_h^t to y_h^t if and only if $u_h(x_h^t, c_h^{t+1}) > u_h(y_h^t, c_h^{t+1})$,

(iii) the endowments of G^t are some $\tilde{e}_t^t$ such that $\sum_{h \in G^t} \tilde{e}_t^t = \sum_{h \in G^t} e_t^t$,

(iv) G^t is irreducible under the new preferences and endowments.

An allocation c belongs to the *short run core* if for any $s \in T$ there exists some $t \geqslant s$ such that $(c^1, ..., c^t)$ belongs to the core of a t-economy with respect to c_t^{t+1}.

We will use the following notation: the set of competitive equilibrium allocations (E), the core (C), the bounded core (BC), the set of Pareto optimal allocations (PO), the set of short run Pareto optimal allocations

(SPO), the bounded perturbation core (BPC), and the short run core (SC). We have the following inclusion relations among these sets:

PROPOSITION 1. (i) $C \subset BC$, (ii) $C \subset PO \subset SPO$, (iii) $E \subset SC \subset (BC \cap SPO)$, (iv) $BPC \subset (BC \cap SPO)$.

Proof. See Chae [2] for (i), (ii), and (iii). To show (iv), suppose $c \in BPC$. Since $\sum_{h \in S} e_h \notin F_S^*(c)$ for any coalition S, one has $\sum_{h \in S} e_h \notin F_S^*(c) = F_S(c)$ for any finite coalition S in particular. Thus $c \in BC$. Also, since $\sum_{h \in S} e_h \notin F_S^*(c)$ for any coalition S, one has $\sum_{h \in G} e_h \notin F_G^*(c)$ in particular. Thus $c \in SPO$. ∎

Chae [2] shows that (i), (ii), and (iii) hold in a general model where the set of households has both atomic and nonatomic parts. He also shows that $E = SC$ if G is nonatomic.

Let $S^t = S \cap G^t$ and $S_t = S^1 \cup \cdots \cup S^t$ for any coalition S. On the overlap of the set of competitive equilibrium allocations with the core, we have the following result which strengthens the results of Chae [2] and Esteban and Millán [5]:

THEOREM 1. *If a competitive equilibrium* (p, c) *satisfies* $\liminf_{t \to \infty} p^t \sum_{h \in G^t} e_h^t = 0$, *then* c *is in the core.*

Proof. Suppose $c \notin C$. Then there exist some coalition S and assignment x such that (i) $u_h(x_h) \geqslant u_h(c_h)$ for all $h \in S$, (ii) $u_h(x_h) > u_h(c_h)$ for some $h \in S$, and (iii) $\sum_{h \in S} x_h = \sum_{h \in S} e_h$.

Since (p, c) is a competitive equilibrium, (i) implies $px_h \geqslant pc_h = pe_h$ for all $h \in S$, and thus

$$p^t\left(\sum_{h \in S^t} x_h^t - \sum_{h \in S^t} e_h^t\right) + p^{t+1}\left(\sum_{h \in S^t} x_h^{t+1} - \sum_{h \in S^t} e_h^{t+1}\right) \geqslant 0. \tag{1}$$

But, from (iii), one has $\sum_{h \in S^t} x_h^t - \sum_{h \in S^t} e_h^t = -(\sum_{h \in S^{t-1}} x_h^t - \sum_{h \in S^{t-1}} e_h^t)$ for $t = 2, 3, ..., \infty$, and $\sum_{h \in S^1} x_h^1 - \sum_{h \in S^1} e_h^1 = 0$. Thus

$$\begin{aligned} & p^{t+1}\left(\sum_{h \in S^t} x_h^{t+1} - \sum_{h \in S^t} e_h^{t+1}\right) \\ & \geqslant p^t\left(\sum_{h \in S^{t-1}} x_h^t - \sum_{h \in S^{t-1}} e_h^t\right) \\ & \cdots \\ & \geqslant p^2\left(\sum_{h \in S^1} x_h^2 - \sum_{h \in S^1} e_h^2\right) \\ & \geqslant -p^{-1}\left(\sum_{h \in S^1} x_h^1 - \sum_{h \in S^1} e_h^1\right) \\ & = 0. \end{aligned}$$

From (ii), at least one of the above inequalities is strict if t is sufficiently large. Thus there exists some $\delta > 0$ such that $p^t(\sum_{h \in S^{t-1}} x_h^t - \sum_{h \in S^{t-1}} e_h^t) \geqslant \delta$ for any t sufficiently large. Since $p^t(\sum_{h \in S^{t-1}} x_h^t - \sum_{h \in S^{t-1}} e_h^t) \leqslant p^t \sum_{h \in S^t} e_h^t \leqslant p^t \sum_{h \in G^t} e_t^t$, one also has $p^t \sum_{h \in G^t} e_h^t \geqslant \delta$ for all sufficiently large t. Therefore, one cannot have $\liminf_{t \to \infty} p^t \sum_{h \in G^t} e_t^t = 0$. ∎

3. Homogeneous Generation

We will now investigate a special case of the model of the previous section. Consider the following assumptions:

Homogeneity: Households in the same generation have the same preferences and endowments.

Strict Quasiconcavity: The utility functions of the households are strictly concave.

Uniformity: All generations have the same number, n, of households.

Parametrizing the model with the above three assumptions by n, we will denote the set of competitive equilibrium allocations by $E(n)$, the bounded perturbation core by $BPC(n)$, and the short run core by $SC(n)$. Note that allocations in $E(n)$, $BPC(n)$, or $SC(n)$ are short run Pareto optimal allocations. Since the utility functions are strictly quasiconcave, all households in the same generation have the same consumption in any of these sets.

Denote the consumptions of the representative households in $E(n)$ by $E(n)/n$. More precisely, $E(n)/n$ is the consumption profile (in $E(n)$) of the subset of households formed by selecting one household from each generation. The sets $BPC(n)/n$, $SC(n)/n$, and $SSC(n)/n$ can be defined similarly. Notice that $E(n)/n = E(1)$ for any n under the three assumptions above.

Theorem 2. *Under the assumptions of homogeneity, strict quasiconcavity, and uniformity,* $\bigcap_{n=1}^{\infty} BPC(n)/n = E(1)$.

The theorem is essentially the result of Aliprantis and Burkinshaw [1, Theorem 4.2] corrected for an error. The error is corrected by assuming strict quasiconcavity of preferences rather than quasiconcavity. No separate proof is necessary, for their proof is valid with the correction. We will demonstrate below that the theorem can be interpreted as a discrete version of Chae's short run core equivalence theorem for the special case where the above three assumptions are satisfied. For this special case, the *BPC* is equal to a simplified version of the *SC*, which is defined as follows: Define the natural t-economy with respect to c_t^{t+1} as a t-economy with respect to c_t^{t+1} where $\tilde{e}_t^t = e_t^t$. Also, define the natural short run core as the set of allocations c such that for any $s = 1, 2, ..., \infty$, there exists some $t \geqslant s$

such that $(c^1, ..., c^t)$ belongs to the core of the natural t-economy with respect to c_t^{t+1}. Denote the natural short run core by $NSC(n)$. By definition, $NSC(n) \subset SC(n)$.

THEOREM 3. $BPC(n) = NSC(n)$.

Proof. We will first prove $NSC(n) \subset BPC(n)$. Suppose $c \in NSC(n)$ but $c \notin BPC(n)$. Then there exist some coalition S and assignment x such that

$$\sum_{h \in S} x_h = \sum_{h \in S} e_h,$$
$$u_h(x_h) \geqslant u_h(c_h) \qquad \text{for all} \quad h \in S,$$
$$u_h(x_h) > u_h(c_h) \qquad \text{for some} \quad h \in S,$$

and

$$x_h = c_h \qquad \text{for all but a finite number of } h \in S.$$

Here S cannot be finite, for $NSC(n)$ is a subset of the bounded core. Thus there exists some s such that if $t \geqslant s$ then $x_h = c_h$ for all $h \in S^t$ $(= S \cap G^t)$. In the natural t-economy

$$\sum_{h \in S_{t-1}} x_h + \sum_{h \in S^t} x_h^t = \sum_{h \in S_{t-1}} e_h + \sum_{h \in S^t} e_h^t,$$
$$u_h(x_h) \geqslant u_h(c_h) \qquad \text{for all} \quad h \in S_{t-1},$$
$$u_h(x_h^t, c_h^{t+1}) = u_h(c_h^t, c_h^{t+1}) \qquad \text{for all} \quad h \in S^t,$$

and

$$u_h(x_h) > u_h(c_h) \qquad \text{for some} \quad h \in S_{t-1}.$$

Thus $c \notin NSC(n)$.

We will now show that $BPC(n) \subset NSC(n)$. Suppose $c \in BPC(n)$ but $c \notin NSC(n)$. Then there exists some period s such that for any period $t \geqslant s$, c does not belong to the core of the natural t-economy. That is, there exist some coalition S and assignment $(x^1, ..., x^t)$ such that

$$\sum_{h \in S_{t-1}} x_h + \sum_{h \in S^t} x_h^t = \sum_{h \in S_{t-1}} e_h + \sum_{h \in S^t} e_h^t,$$
$$u_h(x_h) \geqslant u_h(c_h) \qquad \text{for all} \quad h \in S_{t-1},$$
$$u_h(x_h^t, c_h^{t+1}) \geqslant u_h(c_h^t, c_h^{t+1}) \qquad \text{for all} \quad h \in S^t;$$

and

$$u_h(x_h) > u_h(c_h) \qquad \text{for some} \quad h \in S_{t-1}$$

or

$$u_h(x_h^t, c_h^{t+1}) > u_h(c_h^t, c_h^{t+1}) \qquad \text{for some} \quad h \in S^t.$$

Since $BPC(n)$ is a subset of the bounded core, S^t cannot be empty. Let m ($\leq n$) be the number of households in S^t. Let G_m^τ be a subset of G^τ consisting of m households, and $\tilde{S} = S \cup (\bigcup_{\tau=t+1}^{\infty} G_m^\tau)$. Also, let $x^\tau = c^\tau$ for any $\tau > t$. Then $\sum_{h \in \tilde{S}} e_h = \sum_{h \in \tilde{S}} x_h \in F_{\tilde{S}}(c)$. Therefore, $c \notin BPC(n)$. ∎

Using Theorem 3, Theorem 2 can be rewritten as $\bigcap_{n=1}^{\infty} NSC(n)/n = E(1)$. Let us now relate this with the result of Chae [2, Theorem 3.1]. He shows that $SC = E$ for an economy with a continuum of agents. In general, one has $NSC \subset SC$, and thus $NSC \subset E$ for the continuum economy. In the special case where each generation consists of a continuum of homogeneous households who have strictly concave utility functions, one can easily verify $NSC = E$ following the steps used in Chae's proof of $SC = E$. Theorem 2 is the discrete version of this result for the special case. The result, however, is not general. We will demonstrate below that Theorem 2 does not hold if either the homogeneity or the strict concavity assumption is violated.

EXAMPLE 1 (Violation of Homogeneity). In each generation, there are n households of each of two types of households, A, B, with the same utility function $u_h(x, y) = v(x) + v(y)$, where v is a continuous, increasing, and strictly concave function, but different endowments $e_A = (a, b)$, $e_B = (b, a)$, where $a > b$. Let $d = (a + b)/2$. In the unique competitive equilibrium, one has $c_h = (d, d)$ for any h. But $c \notin BPC$, for it can be improved upon by the coalition of type A agents with the consumption profile which is the same as c except that $c_h = (a, d)$ for any h in G^1.

EXAMPLE 2 (Violaton of Strict Quasiconcavity). In each generation, there are $2n$ households with the same linear utility function $u_h(x, y) = x + y$ and the same endowments $e_h = (d, d)$. Let a, b be nonnegative numbers such that $a + b = 2d$ and $a > b$. An allocation c where one half of each generation has the consumption (a, b) and the other half (b, a) is a competitive equilibrium allocation. But c can be improved upon by the coalition of those who consume (b, a) with the consumption profile which is the same as c except that $c_h = (d, a)$ for any $h \in G^1$.

REFERENCES

1. C. D. ALIPRANTIS AND O. BURKINSHAW, An overlapping generations model core equivalence theorem, *J. Econ. Theory* **50** (1990), 362–380.
2. S. CHAE, Short run core equivalence in an overlapping generations model, *J. Econ. Theory* **43** (1987), 170–183.

3. S. Chae and J. Esteban, The bounded core of an overlapping generations economy, *Int. Econ. Rev.* **30** (1989), 510–525.
4. J. Esteban, A characterization of the core in overlapping generations economies, *J. Econ. Theory* **39** (1986), 439–456.
5. J. Esteban and T. Millán, Competitive equilibria and the core of overlapping generations economies, *J. Econ. Theory* **50** (1990), 155–174.
6. K. Hendricks, K. Judd, and D. Kovenock, A note on the core of the overlapping generations model, *Econ. Lett.* **6** (1980), 95–97.
7. D. Kovenock, A second note on the core of the overlapping generations model, *Econ. Lett.* **14** (1984), 101–106.
8. P. Samuelson, An exact consumption-loan model of interest with or without the social contrivance of money, *J. Polit. Econ.* **66** (1958), 467–482.