JOURNAL OF ECONOMIC THEORY 39, 439–456 (1986)

Notes, Comments, and Letters to the Editor

A Characterization of the Core in Overlapping-Generations Economies*

J. ESTEBAN

Instituto de Análisis Económico, CSIC and Departament de Teoria Econòmica, Universitat Autònoma de Barcelona, Bellaterra-Barcelona, Spain

Received February 11, 1984; revised June 18, 1985

In this paper we characterize the set of consumption allocations that belong to the core and show that for the case of one agent per generation and one good no monetary equilibrium is in the core. For economies with many goods we give bounds on the value of money transactions for competitive equilibria to belong to the core. *Journal of Economic Literature* Classification Numbers: 021, 023. © 1986 Academic Press, Inc.

1. INTRODUCTION

In this paper we analyse overlapping-generations models from a game theoretical point of view. Specifically we characterise the set of consumption allocations that are in the core of the economy.[1] Since Samuelson's [7] contribution overlapping-generations models have been seen as the appropriate framework in which the "social contrivance of money" can be emphasized. The introduction of fiat money permits the implementations of Pareto-optimum allocations that could not have been reached in barter economies. The question arises of whether the institution of fiat money is a stable one, i.e., will not be refused by some agents. It turns out that in economies with one good per period no Pareto-optimal "monetary" allocation can be in the core. This result is no longer true for economies with more than one good. With many goods we find that there

* This paper owes much to the discussions held at a workshop on overlapping-generations models with X. Calsamiglia, I. Fradera, A. Mas-Colell, T. Millán, J. Oliu, and C. Ponsati as regular co-participants. Responsibility for errors is solely mine. Financial support from the Instituto de Estudios Fiscales, Madrid, is gratefully acknowledged.

[1] In independent research Chae [3] characterizes the core and bounded core in a similar model but with a continuum of agents.

0022-0531/86 $3.00

exist bounds to the value of monetary transactions beyond which Pareto optimal monetary equilibria do not belong to the core.

The paper is organized as follows. After introducing notation and some basic definitions we define the notions of blocking coalition and the core in Section 3. The next section is devoted to the characterization of the core in one-good and n-good economies. Section 5 introduces criteria for identifying those equilibria (either barter or monetary) that are in the core. In Section 6 we provide examples of economies with an empty core and obtain the set of endowment allocation sequences for which the core is non-empty. Finally, in Section 7 we prove that for all Pareto-optimum allocations that can be implemented as monetary equilibria there exists a feasible redistribution of endowments such that they do belong to the core and still can be implemented as monetary equilibria.

2. Notation and Definitions

The economy we shall analyse is similar to the one described in Balasko and Shell [1]. There are n perishable commodities available at every date and there is no storage of commodities nor any other production activity. One consumer is born at each date, t, and lives for two periods, t and $t+1$. Each generation is denoted by its birth date, t. At the beginning of this economy there exists a generation previously born at $t=0$.

Let $c_t^{t+s,i}$ be the consumption of commodity i $(i=1, 2,..., n)$ in period $t+s$ $(s=0, 1)$ by consumer t $(t=0, 1, 2,...)$. Preferences of consumer t can be represented by a utility function $u_t: R_+^{2n} \to R_+$, for $t=1, 2,...$, and $u_0: R_+^n \to R_+$ for consumer $t=0$, i.e.,

$$u_t(c_t), \qquad t=0, 1, 2,...,$$

where $c_0 = c_0^1 = (c_0^{1,1}, c_0^{1,2},..., c_0^{1,n}) \in R_+^n$, for $t=0$, and $c_t = (c_t^t, c_t^{t+1}) = (c_t^{t,1},..., c_t^{t,n}, c_t^{t+1,1},..., c_t^{t+1,n}) \in R_+^{2n}$, for $t=1, 2...$.

Assumption 1. u_t is strictly monotonic increasing in each argument, $t=0, 1, 2,...$.

Each consumer has positive endowments of the goods during his lifetime,

$$w_0 = w_0^1 = (w_0^{1,1}, w_0^{1,2},..., w_0^{1,n}) \in R_+^n \qquad \text{for} \quad t=0,$$

and

$$w_t = (w_t^t, w_t^{t+1}) = (w_t^{t,1},..., w_t^{t,n}, w_t^{t+1,1},..., w_t^{t+1,n}) \in R_+^{2n}$$
$$\text{for} \quad t=1, 2,...$$

In this economy a consumption allocation is a sequence of consumption vectors specifying for each date the consumption of each agent existing in

the economy. A consumption allocation is then a sequence $c = (c_0, c_1, \ldots)$. We denote by C the set of all such sequences and by W the set of positive individual endowments sequences, $w = (w_0, w_1, \ldots)$.

DEFINITION 1. A consumption allocation c is feasible if for all t, $t = 1, 2, \ldots$,

$$c^t_{t-1} + c^t_t \leqslant w^t_{t-1} + w^t_t.$$

Following Balasko and Shell [1] we shall now introduce the following definitions, concerning the optimality of consumption allocations.

DEFINITION 2. The allocation $c = (c_0, c_1, \ldots, c_t, \ldots)$ is Pareto-optimal (PO) if there is no $\bar{c} = (\bar{c}_0, \bar{c}_1, \ldots, \bar{c}_t, \ldots) \in C$ such that

(i) both c and $\bar{c}$ are feasible; and

(ii) $u_t(\bar{c}_t) \geqslant u_t(c_t)$ for $t = 0, 1, 2, \ldots$, with at least one strict inequality.

DEFINITION 3. The allocation $c = (c_0, c_1, \ldots, c_t, \ldots)$ is *weakly Pareto-optimal* (WPO) if there is no $\bar{c} = (\bar{c}_0, \bar{c}_1, \ldots, \bar{c}_t, \ldots) \in C$ such that

(i) both c and $\bar{c}$ are feasible;

(ii) $\bar{c}_t = c_t$ except for a finite number of t; and

(iii) $u_t(\bar{c}_t) \geqslant u_t(c_t)$ for $t = 0, 1, 2, \ldots$, with at least one strict inequality.

In the next section we shall make extensive use of the following definitions, which turn out to play a crucial role in characterizing the core of this economy.

DEFINITION 4. We shall say that an allocation c is *time-wise* Pareto superior (TWPS) with respect to allocation $\bar{c}$ if

(i) $u_t(c^t_t, c^{t+1}_t) \geqslant u_t(\bar{c}^t_t, c^{t+1}_t)$ and

(ii) $u_t(c^t_t, c^{t+1}_t) \geqslant u_t(c^t_t, \bar{c}^{t+1}_t)$, for $t = 1, 2, \ldots$, and $u_0(c^1_0) \geqslant u_0(\bar{c}^1_0)$.

DEFINITION 5. The allocation c is *time-wise individually rational* (TIR) if t is TWPS with respect to the endowments allocations sequence $w \in W$.

DEFINITION 6. The allocation c is *individually rational* if it is Pareto superior with respect to the endowments sequence $w \in W$.

3. The Bargaining Problem

Consumption allocations can be viewed as the outcome of quantity bargaining. In this economy only two agents coexist at each period and their resources cannot be transferred from one period to the next. In some respects, this problem looks very much like finding a sequence of solutions to static, temporarily unconnected, bargaining problems with two agents and given joint endowments at every date. Yet one of the two agents will survive until the end of the next period and this opens new possibilities to the bargaining problem that are not conceivable in the static analog. More specifically, agent t may obtain from young agent $t+1$ extra resources if he can persuade him that next period he will in turn be compensated by agent $t+2$ and so on. This case is illustrated by the "chocolate bar" example due to Shell [8]. The possibility of such trades is what may prevent competitive barter allocations from being Pareto optimal. This has given rise to the recent body of literature[2] emphasizing the role of fiat money (as a store of value) in implementing Pareto-optimal allocations in infinite-horizon overlapping-generations economies.

Let us assume that there is full information, that is, every agent knows the preferences and endowments of all future generations. A trade proposal from agent 0 to agent 1 consists of a sequence of net trades. The proposed consumption allocation will not be acceptable to agent 1 if there exists a subset of future agents that by forming a coalition can improve upon this allocation. Let agent f be the first member of this blocking coalition. Given full information, agent $f-1$ would not accept the proposed consumption allocation because he knows that it will not be accepted by agent f. The sequential structure of the model imposes that only those allocations that are acceptable for all unborn generations can be acceptable for any two coexisting agents.[3] The main point is that even though the whole sequence of contracts cannot be "signed" today, any two coexisting rational agents will behave as if these contracts could actually be agreed upon by all agents. It is in this sense that we find it justifiable to talk of "coalitions."

Definition 7. A *coalition* is a non-empty connected subset S of the set of all agents $(0, 1, 2,...)$, that is, if t and $t+k$ belong to S, $t+r$ belongs to S too, $r = 1, 2,..., k-1$.

[2] See, for instance, Balasko and Shell [1], Cass, Okuno, and Zilcha [2], Millán [6], and Wallace [9].

[3] This behaviour can be related to the "absence of trust" analysed by Gale [4]. Here, agents do not violate future contracts undertaken by themselves, but realize that some contracts might be unacceptable to unborn agents.

DEFINITION 8. An allocation $\bar{c}$ is *blocked by coalition S* if there exists another allocation c such that

(i) it is feasible for coalition S, i.e., $c_{t-1}^t + c_t^t = w_{t-1}^t + w_t^t$ for all $t \in S$, and

(ii) $u_t(c_t) \geqslant u_t(\bar{c}_t)$ for all $t \in S$, with at least strict inequality for agent f, $f = \min\{t/t \in S\}$.

DEFINITION 9. The *core* of an economy is the set of all allocations that are feasible and not blocked by any coalition.

4. THE CORE IN OVERLAPPING-GENERATIONS ECONOMIES

PROPOSITION 1. *Let c be a consumption allocation in the core. Then, allocation c satisfies the following properties*:

(i) *Pareto optimality*;

(ii) *individual rationality*; *and*

(iii) *time-wise individual rationality.*

Proof. Conditions (i) and (ii) are rather obvious and need no further comments. Let us concentrate on property (iii).

Assume that the consumption allocation c does not satisfy property (iii), that is, for some agent r we have that either

$$u_r(w_r^r, c_r^{r+1}) > u_r(c_r^r, c_r^{r+1}) \tag{1}$$

or

$$u_r(c_r^r, w_r^{r+1}) > u_r(c_r^r, c_r^{r+1}). \tag{2}$$

Let us consider first inequality (1). In that case, the coalition formed by all $t \geqslant r$ with the allocation x,

$$x_t = c_t \qquad \text{for all} \quad t \neq r-1, r,$$

$$x_{r-1} = (c_{r-1}^{r-1}, w_{r-1}^r),$$

and

$$x_r = (w_r^r, c_r^{r+1}),$$

will block allocation c. Allocation x is feasible and is strictly preferred by the first member of the blocking coalition. Thus, the consumption allocation c cannot belong to the core.

Consider now inequality (2). In such case, the coalition formed by all $t \leqslant r$ with the allocation x,

$$x_t = c_t \quad \text{for all} \quad t \neq r,\ r+1,$$
$$x_r = (c_r^r,\ w_r^{r+1}), \quad \text{and} \quad x_{r+1} = (w_{r+1}^{r+1},\ c_{r+1}^{r+2}),$$

will block allocation c. Q.E.D.

COROLLARY 1. *Let there be one commodity per period, $n = 1$. Then the core of this economy only contains the initial endowments allocation sequence if it is Pareto optimal otherwise it is empty.*[4]

Proof. By Proposition 1 an allocation is in the core only if it is TIR. It is immediate that for $n = 1$, the only feasible allocation that is TIR is the initial endowments allocation itself. Thus, the core is empty except when the initial endowments allocation sequence w is PO. Q.E.D.

Assumption 2. u_t is *weakly separable* in the consumption bundles of each period, i.e., $u_t(c_t^t,\ c_t^{t+1}) = \varphi_t(v_t^t(c_t^t),\ v_t^{t+1}(c_t^{t+1}))$, for $t = 1, 2, 3, \ldots$.

PROPOSITION 2. *Let preferences satisfy Assumptions* 1 *and* 2. *A Pareto-optimal allocation c is in the core if and only if it is time-wise individually rational.*

Proof. That time-wise individual rationality is necessary has already been proven in Proposition 1. Let us prove sufficiency. We can start by noting that by Assumption 2 all TIR allocations are also individually rational. Let us consider now an allocation c that is PO and TIR. Assume that there is a coalition S with an allocation x blocking c. Let f be the first member of this blocking coalition. Since allocation c is TIR,

$$u_f(c_t^t,\ c_f^{f+1}) \geqslant u_f(w_f^f,\ c_f^{f+1}).$$

Being the first member of coalition S, the following inequality must hold:

$$u_f(w_f^f,\ x_f^{f+1}) > u_f(c_f^f,\ c_f^{f+1}).$$

By Assumptions 1 and 2, these inequalities imply that

$$u_f(c_f^f,\ x_f^{f+1}) > u_f(w_f^f,\ x_f^{f+1}) > u_f(c_f^f,\ c_f^{f+1}).$$

[4] A similar result has been obtained by Hendricks, Judd, and Kovenock [5].

For finite coalitions and following the same steps as before, we have for the last member l,

$$u_l(x_l^l, c_l^{l+1}) \geqslant u_l(x_l^l, w_l^{l+1}) \geqslant u_l(c_l^l, c_l^{l+1}).$$

Thus, allocation y such that

$$\begin{aligned} &y_t = c_t && \text{for all} \quad t < f \text{ and } t > l, \\ &y_t = x_t && \text{for all} \quad t, f < t < l, \\ &y_f = (c_f^f, x_f^{f+1}), && \text{and} \qquad y_l = (x_l^l, c_l^{l+1}) \end{aligned}$$

is feasible and $u_t(y_t) \geqslant u_t(c_t)$ with strict inequality at least for agent f. We obtain a contradiction with the hypothesis that allocation c is PO. The extension to the case of coalitions with infinite agents is straightforward.

Q.E.D.

5. Equilibrium Allocations and the Core of the Economy

Let us first introduce further restrictions on consumer preferences.

Assumption 3. u_t has strictly positive first-order partial derivatives and is strictly quasi-concave, $t = 0, 1, 2, \ldots$.

Consumers can buy or sell at current and perfectly foreseen future market prices. Let $p^{t,i}$ denote the price of commodity i $(i = 1, \ldots, n)$ in period t $(t = 1, 2, \ldots)$, p^t denote the vector $(p^{t,1}, \ldots, p^{t,n}) \in R^n_{++}$, and p the price sequence $(p^1, p^2, \ldots)$. We shall normalize by fixing $p^{1,1} = 1$. Let P be the set of such sequences of present prices, $P = \{p / p^{1,1} = 1\}$.

Each consumer choses his lifetime consumption pattern by solving

$$\begin{aligned} &\text{Max } u_0(c_0^1) \\ &\text{subject to } p_0^1 \cdot c_0^1 \leqslant p_0^1 \cdot w_0^1 + m \qquad \text{for} \quad t = 0, \end{aligned}$$

where m is a finite real number; and

$$\begin{aligned} &\text{Max } u_t(c_t^t, c_t^{t+1}) \\ &\text{subject to } p^t \cdot c_t^t + p^{t+1} \cdot c_t^{t+1} \leqslant p^t \cdot w_t^t + p^{t+1} \cdot w_t^{t+1}. \end{aligned}$$

We shall denote by $f_t(p^t, p^{t+1}, w_t)$ the demand vector resulting from agent's t, $t = 1, 2, \ldots$, maximization program. Similarly, $f_0^1(p^1, w_0^1, m)$ is the demand vector corresponding to agent $t = 0$.

Definition 10. Let $w \in W$ be a sequence of positive endowments and m

a finite real number. A *competitive equilibrium* associated with $w \in W$ is a real number m, a sequence of strictly positive commodity prices $p \in P$, and a consumption allocation sequence $c \in C$ such that

(i) $c_t = f_t(p^t, p^{t+1}, w_t)$, $t = 1, 2, \ldots$, and $c_0^1 = f_0^1(p^1, w_0^1, m)$ for $t = 0$; and

(ii) $f_{t-1}^t(p^{t-1}, p^t, w_{t-1}) + f_t^t(p^t, p^{t+1}, w_t) = w_{t-1}^t + w_t^t$, $t = 2, 3, \ldots$, and $f_0^1(p^1, w_0^1, m) + f_1^1(p^1, p^2, w_1) = w_0^1 + w_1^1$.

We shall say that an equilibrium is a "barter" equilibrium when $m = 0$ and that it is a "monetary" equilibrium otherwise.

By adding the budget constraints it is easy to obtain that in equilibrium

$$p^t \cdot (w_t^t - c_t^t) = p^{t+1} \cdot (c_t^{t+1} - w_t^{t+1}) = m, \qquad t = 1, 2, \ldots . \tag{3}$$

We can thus interpret m as the constant nominal value of "money" purchases (normalized with respect to $p^{1,1}$) associated with this equilibrium. From another point of view we can say that m is the nominal amount of "money" we have to endow with agent $t = 0$ in order to sustain this consumption allocation as a "monetary" equilibrium, together with the market price sequence p.

The existence of competitive equilibria has already been proven by Balasko and Shell [5][5] as well as the proposition that competitive allocations are weakly Pareto optimal. For given $w \in W$ we shall have in general different equilibria associated with different values for m and price sequences $p \in P$. Moreover, for $m = 0$, there always exists an equilibrium price sequence $p \in P$ associated with every sequence of endowments $w \in W$ [1, Proposition 3.10].

Taking into account Assumption 2, for a competitive equilibrium m and price sequence p, the sequence of equilibrium consumption allocations can be obtained from the following maximization program:

$$\text{maximize } v_t^t(c_t^t), \text{ subject to } p^t \cdot c_t^t \leqslant p^t \cdot w_t^t - m,$$

and

$$\text{maximize } v_t^{t+1}(c_t^{t+1}), \text{ subject to } p^{t+1} \cdot c_t^{t+1} \leqslant p^{t+1} \cdot w_t^{t+1} + m, \qquad t = 1, 2, \ldots .$$

Let us define the indirect utility function of agent t as

$$u_t = \varphi_t(h_t^t(p^t, p^t \cdot w_t^t - m), h_t^{t+1}(p^{t+1}, p^{t+1} \cdot w_t^{t+1} + m)). \tag{4}$$

[5] Balasko and Shell [1] consider barter equilibria only. Yet the proof can be extended to monetary equilibria, as in Millán [6].

By Assumption 3 h_t^t is strictly decreasing and h_t^{t+1} is strictly increasing in m.

If agent t remains at his initial endowments he obtains a utility level

$$u_t = \varphi_t(v_t^t(w_t^t), v_t^{t+1}(w_t^{t+1})).$$

Let us now consider the following maximization program:

$$\begin{aligned} &\text{maximize } p^t \cdot w_t^t - p^t c_t^t \\ &\quad \text{subject to } v_t^t(c_t^t) \geqslant v_t^t(w_t^t) \qquad \text{for} \quad t = 1, 2, \ldots, \end{aligned}$$

and

$$\begin{aligned} &\text{maximize } p^{t+1} w_t^{t+1} - p^{t+1} c_t^{t+1} \\ &\quad \text{subject to } v_t^{t+1}(c_t^{t+1}) \geqslant v_t^{t+1}(w_t^{t+1}) \qquad \text{for} \quad t = 0, 1, 2, \ldots . \end{aligned}$$

Let the solution to this program be the sequence $\bar{c}$. We can now define

$$b_t^t = p^t \cdot w_t^t - p^t \cdot \bar{c}_t^t(p^t, w_t^t)$$

and

$$b_t^{t+1} = p^{t+1} \cdot w_t^{t+1} - p^{t+1} \cdot \bar{c}_t^{t+1}(p^{t+1}, w_t^{t+1}).$$

By Assumption 3, we have that

$$v_t^{t+s}(w_t^{t+s}) = v_t^{t+s}(\bar{c}_t^{t+s}) \qquad \text{for} \quad s = 0, 1, \text{ and } t = 0, 1, 2, \ldots,$$

and

$$b_t^{t+s} \geqslant 0, \qquad s = 0, 1, \; t = 0, 1, 2, \ldots .$$

The real numbers b_t^{t+s} $(s = 0, 1)$ can be interpreted as the maximum amount of "income" that agent t is willing to pay in order to be allowed access to the market in period $t+s$. We shall thus call b_t^{t+s} the *value of market access* for agent t in period $t+s$, at market prices p^{t+s}.

We can now express the utility level associated with initial endowments as a function of market prices and the value of market access in the periods each egent is alive. Therefore,

$$u_t(w_t) = \varphi_t(h_t^t(p^t, p^t \cdot w_t^t - b_t^t), h_t^{t+1}(p^{t+1}, p^{t+1} \cdot w_t^{t+1} - b_t^{t+1})). \tag{5}$$

For equilibrium consumption allocations, we can restate the definition of *time-wise individual rationality* (TIR) by means of the indirect utility functions (4) and (5).

Let c be an equilibrium consumption sequence relative to the endowment sequence $w \in W$ with associated equilibrium m and price sequence $p \in P$. Then, allocation c is time-wise individually rational if and only if

(i) $m \leqslant b_t^t(p^t)$ for $t = 1, 2, \ldots$, and

(ii) $m \geqslant -b_t^{t+1}(p^{t+1})$ for $t = 0, 1, 2, \ldots$.

PROPOSITION 3. *Let preferences satisfy Assumptions* 2 *and* 3 *and let the endowments sequence be* $w \in W$. *Then, a Pareto-optimal equilibrium consumption allocation sequence* c *is in the core if and only if*

$$-b_t^{t+1}(p^{t+1}) \leqslant m \leqslant b_t^t(p^t) \qquad \textit{for} \quad t = 0, 1, 2, \ldots,$$

where m *and* p *are the associated equilibrium values of nominal money purchases and price sequence.*

Proof. For Pareto-optimum allocations that can be supported by appropriate m and price sequence p, simply apply the above restatement of TIR. Q.E.D.

From Proposition 3 the following corollaries are straightforward.

COROLLARY 2. *Let* c *be a Pareto-optimal consumption allocation sequence that can be supported as a barter competitive equilibrium. Then, allocation* c *is in the core.*[6]

Proof. Barter competitive equilibria are associated with $m = 0$. Thus, the condition established in Proposition 3 is always satisfied. Q.E.D.

COROLLARY 3. *Let* $n = 1$. *Then the core consists only of those Pareto-optimal allocations that are barter competitive equilibria.*

Proof. For $n = 1$, $b_t^t = b_t^{t+1} = 0$ for $t = 0, 1, 2, \ldots$. By Proposition 3 only PO allocations that can be implemented as competitive equilibria with $m = 0$ (barter) can be in the core.

Corollary 3 parallels Corollary 1 but using the characterization of core allocations introduced in Proposition 3.

[6] But Chae [3, Theorem 1] has proven that when blocking coalitions are restricted to have a finite number of members even the non-Pareto-optimal Walrasian allocations belong to the bounded core.

6. The Existence of Core Allocations

We shall now generate examples of empty and non-empty cores by choosing appropriate distributions of the given aggregate endowments between the two agents coexisting in every period. Let us denote by $\bar{w}^t$ the aggregate resources available at date t and by $\bar{w}$ the sequence $(\bar{w}^1, \bar{w}^2,...)$. We shall assume that the sequence $\bar{w}$ is bounded from above and bounded below away from zero. Given a sequence of aggregate endowments $\bar{w}$, we shall denote by $W(\bar{w})$ the set of sequences of individual initial endowments such that

$$w^t_{t-1} + w^t_t = \bar{w}^t \qquad \text{for} \quad t = 1, 2,... .$$

Let $C(\bar{w})$ be the set of consumption allocation sequences satisfying

(i) $c^t_{t+1} + c^t_t = \bar{w}^t$, $t = 1, 2,...$, and

(ii) c_t is uniformly bounded from below by a strictly positive vector y, $t = 0, 1, 2,... .$

Let us consider an economy such that PO allocations exist and that at least one PO consumption allocation sequence $\hat{c}$ belongs to the set $C(\bar{w})$, i.e., $\hat{c} \in C(\bar{w})$. We shall first give an example of economies in which the set of core allocations is empty.

Empty Core

Balasko and Shell [1] have proven that whilst *all* PO allocations are WPO, the contrary is false, i.e., that all WPO are PO. Consider a WPO sequence $\bar{c}$, $\bar{c} \in C(\bar{w})$, that is not PO. Then, for the initial endowments sequence w such that

$$w_t = \bar{c}^t, \qquad t = 0, 1, 2,...,$$

the core is empty. From Definition 3 it is immediate that in this case the set of TIR allocations consists of one element only, namely, the endowment allocation sequence itself. But by assumption this is not PO and hence the core is empty. Let us now give examples of non-empty cores.

Non-empty Core

As before let us restrict to economies for which PO allocations exist and at least one PO consumption sequence $\hat{c} \in C(\bar{w})$. We have a trivial example of an endowment sequence w such that the core is non-empty. This is the case of w such that

$$w_t = \hat{c}_t \qquad \text{for} \quad t = 0, 1, 2,... .$$

Indeed, in that case $\hat{c}$ is TIR and PO. Therefore $\hat{c}$ is in the core.

Let us consider the set $W(\hat{c})$ of the endowments sequences such that

(i) $w \in W(\bar{w})$; and

(ii) $v^t_{t-1}(w^t_{t-1}) \leqslant v^t_{t-1}(\hat{c}^t_{t-1})$ and $v^t_t(w^t_t) \leqslant v^t_t(\hat{c}^t_t)$, $t = 1, 2, \ldots$.

Since $\hat{c} \in C(\bar{w})$, by Assumption 3, the set $W(\hat{c})$ will have more than one element. In fact, this set will contain the intersection of the separating hyperplanes and the set of feasible consumption allocations. Let the endowments sequence $w \in W(\hat{c})$. Then, by construction, $\hat{c}$ is TIR and PO, thus in the core. Moreover, since $\hat{c} \in C(\bar{w})$ and by Assumption 3, there exist endowment sequences $\tilde{w} \in W(\hat{c})$ such that

$$v^t_{t-1}(\hat{c}^t_{t-1}) \geqslant v^t_{t-1}(\tilde{w}^t_{t-1}) \qquad \text{and} \qquad v^t_t(\hat{c}^t_t) \geqslant v^t_t(\tilde{w}^t_t) \text{ for } t = 1, 2, \ldots,$$

with strict inequality at least for some finite number of agents.

7. Core Allocations and Monetary Equilibria

By Corollary 2 we have established that all PO allocations that can be implemented as barter competitive equilibria are in the core. It seems now of interest to analyse whether Pareto-optimal monetary equilibria ($m > 0$) can be in the core. In Section 6 we have shown that given a PO consumption allocation sequence $\hat{c}$ the set $W(\hat{c})$ contains *all* the endowment sequences for which $\hat{c}$ is in the core. Let $\hat{p}$ be the sequence of supporting prices. The problem we are now concerned with is whether there exist an endowment sequence $\hat{w} \in W(\hat{c})$ and a strictly positive finite real number $\hat{m}$ such that $\hat{c}$ is a monetary equilibrium supported by prices $\hat{p}$ and $\hat{m}$ and belongs to the core. Yet, a prior question arises about the implementability of PO allocations as monetary equilibria.

Definition 11. Let $\bar{w}$ be the sequence of aggregate endowments, c a Pareto-optimal allocation, and p the associated supporting prices. A Pareto-optimal allocation c will be said to be *weakly implementable as a monetary competitive equilibrium* (WIME) if there exist a distribution of aggregate endowments $w \in W(\bar{w})$ and a strictly positive finite real number m such that c can be obtained as a monetary competitive equilibrium (Definition 10).

We shall now show that not all PO allocations are WIME. But before going into this result let us mention a proposition due to Balasko and Shell [1] that gives a complete characterization of PO allocations in terms of the associated sequence of supporting prices.

Assumption 4 (Balasko and Shell). Let c be a WPO allocation supported by the price sequence p. Then assume that:

(a) The Gaussian curvature at every point on consumer t's indifference surface through c_t is uniformly bounded from above.

(b) The Gaussian curvature of consumer t's indifference surface through c_t at avery point $y = (y_0,..., y_t,...)$ such that $0 \leqslant y_t^{t+1} \leqslant c_t^{t+1} + c_{t+1}^{t+1}$, for $t = 0, 1,...,$ and $0 \leqslant y_t^t \leqslant c_t^t + c_{t-1}^t$, for $t = 1, 2,...,$ is uniformly bounded away from 0.

(c) There exists a constant P (independent of t) such that

$$0 < P \leqslant \frac{p^{s,i}}{\|(p^t, p^{t+1})\|}$$

for $s = t, t+1$; $i = 1, 2,..., n$; and $t = 1, 2,...$. Furthermore, there exists a constant p' (independent of t) such that

$$0 < P' \leqslant \frac{p^{t,i}}{\|p^t\|}$$

for $i = 1,..., n$; and $t = 1, 2,...$.

(d) The sequence c is bounded from above.

(e) The sequence c is bounded from below by a strictly positive vector.

Their result [1, Proposition 5.6] is the following. Under Assumptions 3 and 4, the WPO consumption allocation sequence c is PO if and only if

$$\sum_t \frac{1}{\|p^t\|} = +\infty. \tag{6}$$

In view of (6), it is obvious that

$$\operatorname*{Lim\,inf}_{t \to \infty} \|p^t\| = 0 \tag{7}$$

is a sufficient condition for Pareto optimality. Yet, no PO consumption allocation satisfying (7) is WIME.

PROPOSITION 4. *Let consumer preferences satisfy Assumptions* 3 *and* 4 *and let c be a PO consumption allocation sequence with supporting prices p. Then, c is weakly implementable as a monetary equilibrium if and only if the*

sequence p is uniformly bounded from below by a strictly positive scalar, λ, i.e.,

$$\operatorname*{Lim\,inf}_{t \to \infty} \|p^t\| \geqslant \lambda > 0.$$

Proof. Consider the sequence c and the associated sequence of supporting hyperplanes defined by

$$p^t \cdot c^t_{t-1} = K^t, \qquad t = 1, 2, \ldots, \tag{8}$$

with $c^t_t = \bar{w}^t - c^t_{t-1}$. This consumption allocation will be WIME if we can find a sequence of initial endowments $w \in W(\bar{w})$ and a strictly positive finite m such that

$$p^t \cdot c^t_{t-1} = p^t \cdot w^t_{t-1} + m, \qquad t = 1, 2, \ldots,$$

that is,

$$p^t \cdot w^t_{t-1} = K^t - m. \tag{9}$$

Therefore our problem is finding a value of m for which the sequence of hyperplanes (9) intersects the set of feasible endowment allocations, i.e.,

$$0 \leqslant w^t_{t-1} \leqslant \bar{w}^t, \qquad w^t_t = \bar{w}^t - w^t_{t-1} \qquad \text{for} \quad t = 1, 2, \ldots .$$

Let $d^t(m)$ be the distance between the hyperplanes defined by (8) and (9), that is,

$$d^t(m) = \frac{m}{\|p^t\|}, \qquad t = 1, 2, \ldots,$$

and δ^t the distance between the hyperplane (8) and the origin, that is,

$$\delta^t = \frac{K^t}{\|p^t\|}, \qquad t = 1, 2, \ldots .$$

The necessary and sufficient condition for the sequence of hyperplanes (9) to intersect the set of feasible endowments allocations (i.e., c be WIME) is that for some $\tilde{m}$, $0 < \varepsilon \leqslant \tilde{m} < +\infty$,

$$d^t(\tilde{m}) < \delta^t$$

and this implies that

$$\tilde{m} < K^t = \delta^t \cdot \|p^t\| \qquad \text{for} \quad t = 1, 2, \ldots .$$

By Assumption 4 ((d) and (e)) δ^t is bounded from above and from below away from 0, so that there exist q and Q such that

$$0 < q \leqslant \operatorname*{Lim\,inf}_{t \to \infty} \delta^t \leqslant \operatorname*{Lim\,sup}_{t \to \infty} \delta^t \leqslant Q < +\infty.$$

Thus,

$$q \cdot \operatorname*{Lim\,inf}_{t \to \infty} \|p^t\| \leqslant \operatorname*{Lim\,inf}_{t \to \infty} K^t \leqslant Q \cdot \operatorname*{Lim\,inf}_{t \to \infty} \|p^t\|.$$

If $\operatorname{Lim\,inf}_{t \to \infty} \|p^t\| = 0$, there is no strictly positive $\tilde{m}$ satisfying the necessary and sufficient condition for c to be WIME.

If $\operatorname{Lim\,inf}_{t \to \infty} \|p^t\| = \lambda > 0$, for any $\tilde{m}$ such that $0 < \tilde{m} \leqslant q \cdot \lambda$ the sequence of hyperplanes $p^t \cdot w^t_{t-1} = K^t - \tilde{m}$ intersects the set of feasible endowment allocations. Therefore, for any endowment allocation $\tilde{w}$ in this intersection, c can be obtained as a monetary equilibrium supported by p and $\tilde{m}$.

Q.E.D.

This is depicted in Fig. 1.

Let us finally examine whether monetary equilibria can belong to the core. In order to give a precise meaning to the discussion of this point let us introduce the following definition.

DEFINITION 12. We shall say that a consumption allocation sequence c is *weakly in the core* (WC) if there exists an endowments sequence $w \in W(\bar{w})$ such that c is in the core.

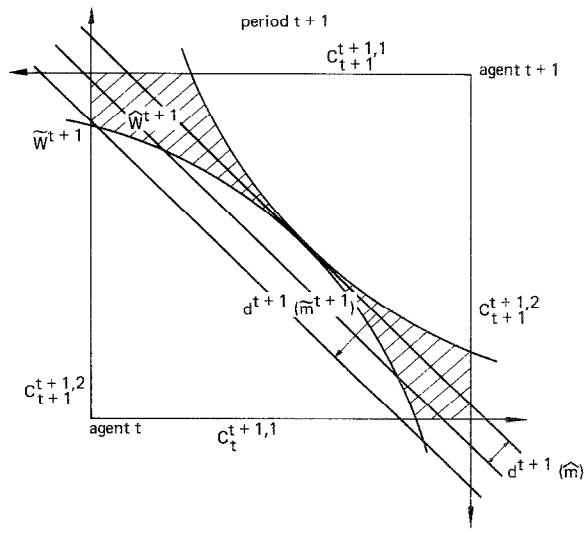

FIGURE 1

PROPOSITION 5. *Under Assumptions* 2, 3, *and* 4, *all PO consumption allocation sequences* $\hat{c}$ *supported by* $\hat{p}$ *that are WIME are also WC. In other words, let* $\hat{c}$ *be WIME for some* $w \in W(\bar{w})$ *and* m, *then there exist* $\hat{w} \in W(\bar{w})$ *and* $\hat{m}$ *such that* $\hat{c}$ *belongs to the core and can be attained as a monetary equilibrium supported with* $\hat{p}$ *and* $\hat{m}$.

Proof. Let $\hat{c}$ be WIME, i.e., for all $m \leqslant q \cdot \lambda$ the intersection of the sequence of hyperplanes $p^t \cdot w_{t-1}^t = K^t - m$, $t = 1, 2,...$, with the set $W(\bar{w})$ is non-empty. In Section 6 we have shown that the set of endowments allocation sequences for which $\hat{c}$ belongs to the core is $W(\hat{c})$. Obviously, $W(\hat{c}) \subset W(\bar{w})$. The point now is whether there exists $\hat{m}$, $0 < \hat{m} \leqslant q \cdot \lambda$, such that the intersection of the sequence of hyperplanes $p^t \cdot w_{t-1}^t = K^t - \hat{m}$ and $W(\hat{c})$ is non-empty. Then for any endowments sequence $\hat{w}$ belonging to this intersection $\hat{c}$ is in the core and can be implemented as a monetary equilibrium supported by $\hat{m}$ and prices $\hat{p}$.

We shall work for convenience with w_t^t, $w_t^t = \bar{w}^t - w_{t-1}^t$. Define the following maximization program:

$$\max_w \hat{p}^t \cdot w_t^t \text{ subject to } v_t^t(w_t^t) \leqslant v_t^t(\hat{c}_t^t) \text{ and } w_t^t \in W(\bar{w})$$

$$\text{for} \quad t = 1, 2,... .$$

Let $\tilde{w}_t^t$, $t = 1, 2,...$, be a solution to this problem and denote by $\tilde{m}^t$

$$\tilde{m}^t = \hat{p}^t \cdot \tilde{w}_t^t - \hat{p}^t \cdot \hat{c}_t^t.$$

Since $\hat{c}$ is in the strict interior of $W(\bar{w})$ (Assumption 4(e)) and preferences are strictly quasi-concave (Assumption 3) we have that $\tilde{m}^t > 0$, $t = 1, 2,...$. Moreover, by Assumption 4(b) there exists some $\gamma > 0$ such that

$$\operatorname*{Lim\,inf}_{t \to \infty} \tilde{m}^t \geqslant \gamma > 0.$$

Therefore, for any $\hat{m}$, $0 < \hat{m} \leqslant \gamma$, the intersection of the sequence of hyperplanes $\hat{p}^t \cdot w_t^t = \hat{p}^t \cdot \hat{c}_t^t + \hat{m}$ with the set $W(\hat{c})$ is non-empty. Q.E.D.

8. CONCLUDING REMARKS

In this paper we have explored the set of core allocations in an overlapping-generations economy and have found that the set of competitive equilibria is not a subset of the core. Moreover, the set of core allocations might be empty. One may think that the core is small because in the economy we have described each period is essentially unconnected with the collateral periods. Indeed, goods are perishable and the allocation arrived

at in any date t imposes no physical restrictions on the allocations that can be agreed upon at any future date. Moreover, since individual utility functions are separable, today's agreements have no influence on the utility level that can be obtained tomorrow. Let us briefly examine the possibility of enlarging the core by diminishing this independence between periods. An obvious way of establishing links across periods is by dropping Assumption 2, i.e., time-separability of utility functions. Yet we have proven in Proposition 1 that TIR is a necessary condition for belonging to the core. Therefore, dispensing with Assumption 2 will not enlarge the core. Another possibility is to drop the assumption that goods are perishable. Under non-perishability, the main difference is that stocks can be carried over from one period to the next. Let us focus on period t. Agent $t-1$'s resources may now be larger than his current endowments, but this is indistinguishable from the case in which agent $t-1$ has more endowments when old. As for agent t, the possibility of storing may reduce but never increase the implicit losses associated with blocking. Thus, the core cannot be larger than that under perishability. It might be interesting to explore the case of productive economies in which some inputs (capital goods) are owned by the old generation and some (labour) by the young one.

Hendricks, Judd, and Kovenock [5] show that in one-good economies if only finite-member coalitions are allowed the core consists of the endowment allocation. For an n-goods economy the core will consist of all those consumption allocation sequences that are WPO and TIR. However, this is not an interesting way of enlarging the core, for at least two reasons. A first objection is the following. Let c and c' be two WPO consumption allocation sequences such that c can be obtained from c' by means of a sequence of Pareto improving transfers. One finds it difficult to accept that allocation c' cannot be blocked by allocation c. Yet, this is the case when we restrict to finite-member coalitions.[7] The second point worth mentioning is that none of the allocations that this assumption permits including in the core is Pareto optimal. Thus, the fact that some Pareto-optimal monetary equilibria might not be in the core remains completely unaffected.

One can argue that in this model agents have some sort of "limited" rationality. Agents find it advantageous to "print their own money." But they should realize that their money will in turn be refused next period. This argument, however, can be reversed in the following way. Agents refuse current money because they realize that it will not be accepted by future generations. Being "generous" with old people today does not contribute anything to induce the next generation to be generous tomorrow. In

[7] Balasko and Shell have proven [1, Lemma 5.4] that if agent t receives a non-zero transfer, so do agents $t+1$, $t+2$,... .

other words, selfish rational agents have no reason for trusting future generations. Therefore, it seems that a promising way for enlarging the core is to introduce institutions that make trust a desirable behaviour for selfish rational agents.

References

1. Y. Balasko and K. Shell, The overlapping generations model. I. The case of pure exchange without money, *J. Econ. Theory* **23** (1980), 281–306.
2. D. Cass, M. Okuno, and I. Zilcha, The role of money in supporting the pareto optimality of competitive equilibrium in consumption-loan type models. *J. Econ. Theory* **20** (1979), 41–80.
3. S. Chae, Bounded core equivalence and core characterization, *J. Econ. Theory*, in press.
4. D. Gale, The core of a monetary economy without trust, *J. Econ. Theory* **19** (1978), 156–491.
5. K. Hendricks, K. Judd, and D. Kovenock, A note on the core of the overlapping generations model, *Econ. Lett.* **6** (1980), 95–97.
6. T. Millán, "On the Existence of Optimal Competitive Equilibria in the Overlapping-Generations Model," Ph.D. thesis, University of Minnesota, 1981.
7. P. A. Samuelson, An exact consumption-loan model of interest with or without the social contrivance of money, *J. Polit. Econ.* **66** (1958), 467–482.
8. K. Shell, Notes on the economics of infinity, *J. Polit. Econ.* **79** (1971), 1002–1011.
9. N. Wallace, The overlapping generations model of fiat money, *in* "Models of Monetary Economies" (J. Kareken and N. Wallace, Eds.), pp. 49–82, Federal Reserve Bank of Minneapolis, Minn., 1980.