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THE BOUNDED CORE OF AN OVERLAPPING GENERATIONS ECONOMY

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The bounded core is the set of allocations which are not improved upon by any bounded set of agents. We submit that the bounded core does not serve as a criterion for the efficiency of allocations in an overlapping generations economy, for it may include allocations which are not even short run Pareto optimal. We also submit that the bounded core is a weak criterion for the robustness of allocations in an overlapping generations economy, for it may not shrink to the set of competitive equilibrium allocations as the individual agents become negligible.

1. INTRODUCTION

In traditional general equilibrium theory, the following propositions rank among the most celebrated ones:

First Welfare Theorem (FWT): A competitive equilibrium allocation is Pareto optimal.

Second Welfare Theorem (SWT): A Pareto optimal allocation is a competitive equilibrium allocation with wealth redistribution.

Core Containment Theorem (CCT): A competitive equilibrium allocation is a core allocation.

Core Equivalence Theorem (CET): If the set of agents is a nonatomic measure space, then a core allocation is a competitive equilibrium allocation.

The FWT and the SWT indicate the efficiency of a market system, while the CCT and the CET indicate the institutional stability of a market system. Efficiency is a necessary condition for institutional stability, that is, a core allocation is Pareto optimal. Thus the CCT implies the FWT in particular.

It has been recognized since Malinvaud (1953) that a market system is efficient in allocating resources over a finite, but not over an infinite time horizon. In the context of an overlapping generations economy, Samuelson (1958) observed that the FWT does not hold. Balasko and Shell (1980) provided conditions under which a competitive allocation is Pareto optimal. They also generalized the FWT and the SWT to an overlapping generations economy, replacing Pareto optimality by short run Pareto optimality.

Regarding the institutional stability of a market system in an infinite horizon economy, there has been some progress both in obtaining conditions under which

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a market allocation is fully stable and in characterizing the limited stability of a market system. From Samuelson's (1958) observation that the FWT does not hold, it is clear that the CCT does not hold either in an overlapping generations economy. Hendricks, Judd and Kovenock (1980) and Kovenock (1984) showed that the core of an overlapping generations economy may even be empty. Esteban (1986) and Kovenock (1984) provided sufficient conditions for a Pareto optimal competitive equilibrium allocation to be in the core of a simple overlapping generations economy. Chae (1987) and Esteban and Millan (1984) provided sufficient conditions for a competitive allocation to be in the core of an overlapping generations economy. Chae (1987) generalized the CCT and the CET to an overlapping generations economy, replacing the core by a short run core.

Chae's short run core is the set of allocations which belong to the cores of a sequence of truncated finite period economies. It is a subset of the bounded core, the set of allocations which are not improved upon by any bounded coalition, as shown by Chae (1987). Thus another generalization of the CCT is obtained by replacing the core by the bounded core. A natural question that arises is whether the short run core is a proper subset of the bounded core. Another interesting question, which can be regarded as a refinement of the previous question, is whether the short run core is a proper subset of the bounded core even if the set of agents is a nonatomic measure space, for in this case the short run core is equal to the set of competitive equilibrium allocations. Since a short run core allocation is short run Pareto optimal, an answer to the first question is easily obtained by observing that a bounded core allocation is not necessarily short run Pareto optimal. Since the same argument holds even if the set of agents is a nonatomic measure space, it also provides an answer to the second question.

In this essay, we will characterize the set of Pareto optimal allocations, the core, the bounded core, the set of short run Pareto optimal allocations, the short run core, and the set of competitive equilibrum allocations for a simple overlapping generations economy. Using this characterization, we show that there exists a bounded core allocation which is not short run Pareto optimal. Using the continuum version of the same economy, we also show that there exists a bounded core allocation which is not a competitive equilibrium allocation even if the set of agents is a nonatomic measure space. We submit that the bounded core does not serve as a criterion for the efficiency of allocations in an overlapping generations economy, for it may include allocations which are not even short run Pareto optimal. We also submit that the bounded core is a weak criterion for the robustness of allocations in an overlapping generations economy, for it may not shrink to the set of competitive equilibrium allocations as the individual agents become negligible.

2. THE MODEL AND RESULTS

Consider a pure exchange, overlapping generations economy. There is one perishable commodity in each period $t = 1, 2, \ldots$. One generation consists of one agent. Generation t ($\neq 2$) has preferences over consumptions during periods t and $t + 1$, represented by the utility function

$$u_t(c_t^t, c_t^{t+1}) = c_t^t + c_t^{t+1}, \quad \text{where} \quad c_t^t, c_t^{t+1} \geq 0,$$

and has endowments $(e_t^t, e_t^{t+1}) = (1, 0)$. Generation 2 has preferences over consumptions during periods 1, 2, and 3, represented by the utility function

$$u_2(c_2^1, c_2^2, c_2^3) = \frac{1}{2} c_2^1 + c_2^2 + c_2^3, \quad \text{where} \quad c_2^1, c_2^2, c_2^3 \geq 0,$$

and has endowments $(e_2^1, e_2^2, e_2^3) = (0, 1, 0)$.

Note incidentally that if we regard the first two periods in the above economy as one period with two commodities, then the economy becomes a more familiar one where each generation lives for two periods, except for the first generation which lives for one period.

We will use the following notation: $c_t = (c_t^t, c_t^{t+1})$ and $e_t = (e_t^t, e_t^{t+1})$ for $t \neq 2$, $c_2 = (c_2^1, c_2^2, c_2^3)$, $e_2 = (e_2^1, e_2^2, e_2^3)$, $c = (c_1, c_2, \ldots)$, and $e = (e_1, e_2, \ldots)$. We may regard c_t as a point which has at most two (if $t \neq 2$) or three (if $t = 2$) nonzero coordinates in the infinite dimensional commodity space. Thus, the expression $\Sigma_{t \in S}\, c_t$ is well defined for any (nonempty) subset S of the set of generations $G = \{1, 2, \ldots\}$.

An assignment c is called an *allocation* if it is feasible, i.e., $\Sigma_{t \in G}\, c_t = \Sigma_{t \in G}\, e_t$. For our economy, this condition can be rewritten as $c_1^1 + c_2^1 = 1$ and $c_{t-1}^t + c_t^t = 1$ for any $t \geq 2$. A nonempty subset S of the set of generations G is called a coalition. If it is bounded, it is called a *bounded coalition*. For our economy, a coalition is bounded if and only if it is finite. (In an economy where the set of agents is a continuum, a bounded coalition need not be finite.) An allocation c is said to be self attainable for coalition S if $\Sigma_{t \in S}\, c_t = \Sigma_{t \in S}\, e_t$. A coalition S is said to improve upon an allocation c if there exists another allocation $\tilde{c}$ which is self attainable for S and which makes someone in S better off without making anyone in S worse off, i.e., $u_t(\tilde{c}_t) \geq u_t(c_t)$ for any $t \in S$ and $u_t(\tilde{c}_t) > u_t(c_t)$ for some $t \in S$.

DEFINITION 1. *An allocation c is said to be*

i) Pareto optimal, *if it is not improved upon by G,*

ii) *a* core *allocation, if it is not improved upon by any coalition,*

iii) *a* bounded core *allocation, if it is not improved upon by any bounded coalition,*

iv) short run Pareto optimal, *if there exists no allocation $\tilde{c}$, which differs from c only in finite periods, such that $u_t(\tilde{c}_t) \geq u_t(c_t)$ for any t and $u_t(\tilde{c}_t) > u_t(c_t)$ for some t,*

v) *a* short run core *allocation, if for any $s \geq 2$, there exists some $t > s$ such that $(c_1, \ldots, c_{t-1}, c_t^t)$ belongs to the core of the t-economy, which is the t period economy where only generations up to t participate and where generation t has endowment e_t^t and utility function $u_t^t(c_t^t) = c_t^t$,*

vi) *a* competitive equilibrium *allocation, if there exists a price system $p = (p^1, p^2, \ldots)$ such that for any t, $pc_t \leq pe_t$ and $u_t(\tilde{c}_t) \leq u_t(c_t)$ for any $\tilde{c}_t$ which satisfies $p\tilde{c}_t \leq pe_t$.*

For the definition of the short run core in a general overlapping generations economy, see Chae (1987). The above definition is equivalent to the general definition for our economy, where the utility functions are intertemporally separable and where there is only one agent in each generation.

Let us denote by PO, C, BC, SPO, SC, and E the set of Pareto optimal allocations, the core, the bounded core, the set of short run Pareto optimal allocations, the short run core, and the set of competitive equilibrium allocations, respectively. Chae (1987) shows that the following relations hold for a general overlapping generations economy: $C \subset BC$, $C \subset PO \subset SPO$, $E \subset SC \subset BC \cap SPO$, and $E = SC$ if the set of agents is a nonatomic measure space. For our economy, these sets are completely characterized as follows:

$$\begin{aligned} SPO &= \{c \in A;\ c_2^1 = 0 \text{ or } c_2^2 = 1\}, \\ BC &= \{c \in A;\ u_t(c_t) \geq 1 \text{ for any } t\}, \\ PO &= \{c \in SPO;\ \text{for any } t \text{ and } \varepsilon > 0, \text{ there exists some } s > t \text{ such that } c_s^s < \varepsilon\}, \\ SC &= E = \{e\}, \text{ and } C = \emptyset, \end{aligned}$$

where A is the set of allocations. The proof of this characterization is deferred to the Appendix. As a by-product of the characterization, we obtain

PROPOSITION 1. *There exists a bounded core allocation which is not short run Pareto optimal.*

PROOF. Let c be an allocation such that $(c_1^1, c_1^2) = (0, 1)$, $(c_2^1, c_2^2, c_2^3) = (1, 0, 1)$, and $(c_t^t, c_t^{t+1}) = (0, 1)$ for any $t \geq 3$. Then $c \in BC$ but $c \notin SPO$. □

The intuition for this result is easily provided. An allocation is the result of an infinite sequence of transactions. When only finite coalitions are allowed to improve upon an allocation, some Pareto improving reallocation among a finite coalition may not take place, for the reallocation deprives the last agent in the coalition of its transfer from the infinite future.

Since $SC \subset BC \cap SPO$, the above result also implies that SC is a proper subset of BC. Furthermore, one can show that SC is a proper subset of $BC \cap SPO$. Let c be an allocation such that $c_1 = (1, 0)$, $c_2 = (0, 1, 1)$, and $c_t = (0, 1)$ if $t \geq 3$. Then $c \in BC \cap SPO$ but $c \notin SC$. The intuition for this example is also easily provided. An allocation, which is not improved upon by any finite coalition and for which there is no Pareto improving reallocation among any finite coalition, may be improved upon by an agent if the agent's planning horizon is truncated.

We will now show that the above discussions remain valid even if the individual agents become negligible. Consider a variant of the above economy which is the same as the above one except for the demographic structure of a generation. In the old economy, there was one agent in each generation. In the new economy, generation t consists of a continuum of identical agents represented as points in the half chosen interval $G^t = [t, t + 1)$, equipped with the Lebesgue measure. The definitions of PO, C, BC, SPO, SC, and E for this economy are straightforward and will not be formally introduced here. We refer inquisitive readers who demand the formal definitions to Chae (1987). These sets are completely characterized as follows:

$$
\begin{aligned}
SPO &= \{c \in A;\ \textstyle\int_2^3 c_h^1\, dh = 0 \text{ or } \int_2^3 c_h^2\, dh = 1\},\\
BC &= \{c \in A;\ u_h(c_h) \geq 1 \text{ for almost all } h \text{ in } G\},\\
PO &= \{c \in SPO; \text{ for any } t \text{ and } \varepsilon > 0, \text{ there exists some } s > t \text{ such that } \textstyle\int_s^{s+1} c_h^s\, dh < \varepsilon\},\\
SC &= E = \{c \in A;\ c_h = e_h \text{ for almost all } h \text{ in } G\}, \text{ and } C = \emptyset.
\end{aligned}
$$

The proof is similar to that for the discrete economy, and will not be presented here. Since $E = SC \subset BC \cap SPO$, the continuum analogue of the example used to prove Proposition 1 can also be used to prove

PROPOSITION 2. *There exists a bounded core allocation which is not a competitive equilibrium allocation even if the set of agents is a nonatomic measure space.*

Of course, one can also prove the above proposition using the continuum analogue of the example used to show the proper containment of SC in $BC \cap SPO$. The intuition provided for either example also explains why the bounded core does not shrink to the set of competitive equilibrium allocations as the individual agents become negligible.

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APPENDIX

PROPOSITION A.1. *An allocation c is short run Pareto optimal if and only if $c_2^1 = 0$ or $c_2^2 = 1$.*

PROOF. An allocation c is short run Pareto optimal if and only if $(c_1, \ldots, c_{t-1}, c_t^t)$ is Pareto optimal in the t-economy for any $t \geq 3$. This condition is satisfied for our economy if and only if the allocation is efficient in the first two periods, i.e., $c_2^1 = 0$ or $c_2^2 = 1$. □

We remark, in particular, that $e \in SPO$ by the above proposition, for $e_2^1 = 0$ (and $e_2^2 = 1$). In the proofs of the subsequent propositions, we will use the following lemma.

LEMMA 1. *Suppose that the endowment allocation e is short run Pareto optimal. An allocation c belongs to the bounded core if and only if $u_t(c_t) \geq u_t(e_t)$ for any t.*

PROOF. (necessity) If $u_t(c_t) < u_t(e_t)$ for some t, then c is improved upon by the coalition $\{t\}$. Thus, $c \notin BC$.

(sufficiency) Suppose $u_t(c_t) \geq u_t(e_t)$ for any t, but there exists some bounded coalition S which improves upon c. Then S also improves upon e, which implies $e \notin SPO$. □

PROPOSITION A.2. *An allocation c belongs to the bounded core if and only if $u_t(c_t) \geq 1$ for any t.*

PROOF. It follows from Lemma 1, for $e \in SPO$ and $u_t(e_t) = 1$ for any t. □

PROPOSITION A.3. *An allocation c is Pareto optimal if and only if $c \in SPO$, and for any t and $\varepsilon > 0$, there exists some $s > t$ such that $c_s^s < \varepsilon$.*

PROOF. One has $PO \subset SPO$ by definition. Thus an allocation in SPO is also in PO if and only if a Pareto improving transfer from the infinite future is impossible. This condition is satisfied for our economy if and only if a constant transfer ε (> 0) is impossible. □

PROPOSITION A.4. $SC = \{e\}$.

PROOF. It will suffice to show that $(e_1, \ldots, e_{t-1}, e_t^t)$ is the only allocation in the core of the t-economy for any $t \geq 3$.

Since $SPO = PO$ and $BC = C$ for a finite time horizon economy, Lemma 1 has its finite time horizon version: When the endowment allocation is Pareto optimal, an allocation belongs to the core if and only if the utility of each agent is equal to the utility from its endowments.

For a t-economy of our model, the endowment allocation is Pareto optimal and is the only allocation which gives to each agent the same utility as its endowments. Thus, the endowment allocation is the only allocation in the core of the t-economy. □

PROPOSITION A.5. $E = \{e\}$.

PROOF. Since $E \subset SC = \{e\}$, it suffices to show $e \in E$. Let $p^t = 1$ for any t, and $p = (p^1, p^2, \ldots)$. Then (p, e) is a competitive equilibrium. □

PROPOSITION A.6. $C = \phi$.

PROOF. Suppose $c \in C$. Then $c_t^t = 1$ for any $t \geq 2$, for otherwise the coalition $\{t, t+1, \ldots\}$ improves upon c. Thus, by Proposition A.3, $c \notin PO$, which is absurd for $C \subset PO$. □

REFERENCES

BALASKO, Y. AND K. SHELL, "The Overlapping Generations Model I," *Journal of Economic Theory* 23 (1980), 281–306.

CHAE, S., "Short Run Core Equivalence in an Overlapping Generations Model," *Journal of Economic Theory* 43 (1987), 170–183.

ESTEBAN, J., "A Characterization of the Core in Overlapping-Generations Economies," *Journal of Economic Theory* 39 (1986), 439–456.

——— AND T. MILLAN, "Competitive Equilibria and the Core of Overlapping-Generations Economies," mimeo, Universitat Autònoma de Barcelona and Universidad Complutense, 1984.

HENDRICKS, K., K. JUDD AND D. KOVENOCK, "A Note on the Core of the Overlapping Generations Model," *Economics Letters* 6 (1980), 95–97.

KOVENOCK, D., "A Second Note on the Core of the Overlapping Generations Model," *Economics Letters* 14 (1984), 101–106.

MALINVAUD, E. "Capital Accumulation and Efficient Allocation of Resources," *Econometrica* 21 (1953), 233–268.

SAMUELSON, P. A., "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money," *Journal of Political Economy* 66 (1958), 467–482.