Title: **OVERLAPPING GENERATIONS MODELS (6.28.3.3.)**

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Summary

The specific feature of the *Overlapping Generations Model* is that new cohorts are born in every period, live for a number of periods and die. Consequently, at any point in time, society is composed of individuals belonging to the different generations that are currently alive. Because of the demographic structure of the population, no trade involving individuals about to die is possible and hence markets *per se* cannot bring about an efficient allocation of the existing resources. This natural restriction on the working of the markets makes the OLG model to behave very differently from the General Equilibrium. Specifically, we have that: i) typically there exists a vast number of equilibria; ii) competitive equilibria may fail to be Pareto efficient; and iii) Pareto efficient competitive equilibria may not belong to the core of the economy. Further, the introduction of intrinsically worthless fiat money induces the appearance of Pareto superior equilibria, otherwise unreachable. Such intergenerational transfers can also be performed by means of taxes and pensions, public debt or other forms of public intervention. These features make clear that the parallelism between the OLG and the GE model is limited. In order to stress the properties specific to OLG economies we start by examining the simplest model possible. This is the canonical OLG model due to Samuelson. In the subsequent sections we present a detailed discussion of the most salient "anomalies" in a much more general model of a pure exchange economy. We examine first the main properties of competitive equilibria: existence, uniqueness and efficiency. Next we analyze the role of fiat money and the conditions for existence and efficiency of monetary equilibria. Finally, we address the issue of the trustworthiness of intergenerational (implicit) agreements. We conclude with some remarks on future research directions.

1. Introduction

David Hume argued that political power could not emanate from a *social contract* because society was not made of butterflies -who are born and die at the same time-, but composed of overlapping cohorts of humans: the unborn generations could not feel obliged by prior social agreements. Indeed we live in a society in which many social or economic contracts, as well as rules and norms of behavior, involve obligations to future, still unborn generations. Accordingly to Plutarch (Licurgus XV), a young man refused to give his seat at the assembly to old Dercilides of Sparta (IV BC), thus violating existing social rules. He argued that, since Dercilides had remained a bachelor, he had not contributed any children who in their turn might one day would give their seat to him when in his old days.

Intergenerational agreements pervade our social life. Parents give education, values and bequests to their off-springs. But, at the same time, they also shift to their children the burden of public debt, fiat money, and pensions. Further, intergenerational interactions transcend the family nucleolus. At a societal level, current investment in physical capital or in R&D spreads its benefits into future generations. However, our present consumption of non-renewable resources (or partially so, including pollution and global warming) is at the expense of the wellbeing of our descendants.

At first sight, this problem does not look essentially different from the standard allocation of resources over competing needs –the core of modern Economics. Market prices reveal the intensity of the needs relative to the abundance of the necessary resources. Resources end up being allocated efficiently by the interdependence of actions taken by egotistic agents. However, in the case of intergenerational arrangements, a part of the concerned individuals are not present to express the intensity of their preferences. Further, even if we could anticipate their preferences and resources, the sequential nature of the population structure imposes restrictions on the feasibility of trades. We cannot conceive direct (implicit) agreements with generations one century ahead of us without the intermediation of the generations in between. A competitive market proper cannot exist by nature.

There are two main ways of modeling the intergenerational allocation of resources. They essentially differ in the way they treat consumers. In one case, consumers are represented by one single, infinitely lived individual. This highly simplified representation has to be understood as a short hand for a more complex model that can be formalized in a number of alternative ways. The competing way of modeling the inter-temporal allocation of resources is the *Overlapping Generations Model* (OLG hereafter). In the OLG model new cohorts are born in every period, live for a number of periods and die. Therefore, at any point in time, society is composed of individuals belonging to the different generations that are currently alive. As it turns out, the kind of interactions or trades that can occur are constrained by the fact that they concern individuals at different points of their life-span.

The first formal model of an economy with overlapping generations is due to Samuelson (1958). In his seminal paper, the highly stylized economy consists of one individual per generation who lives for two periods and only one (perishable) commodity. At each period there are two players only: the young and the old. All possible trades necessarily involve individuals of different, subsequent generations. Suppose that consumers own the entire stock of commodities when young and nothing when old. Clearly, they would be better off by lending part of these commodities to another consumer to be repayed in the second period of his lifetime. Yet, the old consumer he has nothing to offer in exchange. Thus, because of the demographic structure of the population no trade is possible and markets *per se* cannot bring about an efficient allocation of the existing resources. Then, Samuelson goes on to show that if the government introduces valueless fiat money notes, they will attain a strictly positive market price and the economy will shift to a Pareto superior allocation.

In spite of its extreme simplicity this model contains most of the essential features specific to OLG models which make it substantially different from the standard, static General Equilibrium (GE thereafter) model. In the (Arrow-Debreu) GE model, the time horizon is finite and consumers are all contemporary. Consumers are assumed to behave competitively, that is, make plans and trade in markets using exogenous prices as the sole information. Under some fairly general conditions, competitive equilibria exist -in the sense that there are prices for which all individual plans are mutually compatible- and are locally unique. In addition, (First Fundamental Theorem of Welfare Economics) competitive equilibria are Pareto optima. Finally, competitive equilibrium allocations belong to the *core* of the economy.

These results reinforce each other and render the notion of competitive equilibrium logically coherent. The existence of equilibrium warrants that the definition of competitive equilibrium is not void and the local uniqueness makes the notion determinate: the description of the economy is neither unnecessarily stringent, nor too loose. That equilibira are Pareto optimal shows that competitive exchange through markets leads to efficient outcomes. Lastly, the result that the equilibrium allocations belong to the core of the economy shows in a formal way that it does not pay to individuals (or subsets of) to deviate from competitive behavior and seek for arrangements with other consumers, by-passing the market exchanges. Thus, individuals do rightly by behaving accordingly with the competitive conjecture.

In contrast to the results above, in OLG models we shall have that: i) typically there exists a vast number of equilibria; ii) competitive equilibria may fail to be Pareto efficient; and iii) Pareto efficient competitive equilibria may not belong to the core of the economy. Consequently, we have that, in an OLG environment, the description of the economy as an uncoordinated interaction that takes place through markets in which agents act competitively might be insufficient to be able to yield determinate outcomes. In addition, the working of the markets does not seem to be sufficient to warrant the efficient allocation of resources. Finally, the assumption that consumers take decisions conjecturing a competitive environment might be inconsistent with rational behavior.

The arguments we have just presented make it clear that the trade between coexisting generations will have distinct features. The parallelism with the GE model is limited. In order to stress the properties specific to OLG economies we shall start by examining the simplest model possible. This is the canonical OLG model due to Samuelson. In the subsequent sections we shall present a detailed discussion of the most salient "anomalies" in a much more general model of a pure exchange economy. We shall examine first the main properties of competitive equilibria: existence, uniqueness and efficiency. Next we analyze the role of fiat money and the conditions for existence and efficiency of monetary equilibria. Finally, we address the issue of the trustworthiness of intergenerational (implicit) agreements.

2. **Samuelson's canonical example of an OLG model**

We shall consider time divided into a sequence of periods. At each period one consumer is born. We shall denote each consumer by its birthdate. Consumers live for two periods, and die. Therefore, at any given period t there are two consumers alive: the just born young consumer t and the old consumer $t-1$, born in the previous period. There is one commodity only. It is perishable and thus cannot be stored. The young consumers are endowed with an amount *w* and the old ones with *(1-w)*. Endowments are thus stationary.

Let $\left(x_t^t, x_{t+1}^t\right)$ be the amounts of the good consumed by individual t in periods *t* and *t+1* when young and old, respectively. We shall use $x^t \in \Re^2_+$ to denote the lifetime consumption allocation corresponding to individual *t*, $x_t \in \Re^2_+$ be allocation in period t corresponding to the two individuals living at t. Finally $x = \langle x_t \rangle_{t=1}^{\infty}$ will denote the full consumption allocation in periods *t=1,2…*
Individuals share the same preferences over consumption bundles, which are represented by a utility function $u: \Re^2_+ \to \Re$. We shall assume that *u* is strictly increasing, twice differentiable and strictly quasiconcave.

Let p_t and p_{t+1} be the price of the consumption good at dates *t* and *t+1*. We normalize prices to $p_1=1$. We shall denote by *p* the sequence $p = \langle p_t \rangle_{t=1}^{\infty}$.
Consumers wish to maximize their utility subject to the relevant (lifetime) budget constraint, that is

$$\max u\left(x_t^t, x_{t+1}^t\right)$$
$$\textit{subject to } p_t x_t^t + p_{t+1} x_{t+1}^t \leq p_t w + p_{t+1}(1-w). \qquad (1)$$

Under our assumptions on *u* this problem has a unique and interior solution, and is completely characterized by the following first order conditions:

$$p_t x_t^t + p_{t+1} x_{t+1}^t = p_t w + p_{t+1}(1-w). \quad \text{and (2).}$$
$$\frac{\dfrac{fu\left(x^t\right)}{fx_t^t}}{\dfrac{fu\left(x^t\right)}{fx_{t+1}^t}} = \frac{p_t}{p_{t+1}}. \qquad (3)$$

It is plain that the solution is homogeneous of degree zero in prices. The only variable relevant to consumers is the real interest factor

$$R_t = \frac{p_t}{p_{t+1}}. \qquad (4).$$

Notice now that because of the stationariety of preferences and endowments, the demand functions will be stationary as well. Therefore, we can write

$$x_t^t = x_0(R_t) \text{ and } x_{t+1}^t = x_1(R_t). \qquad (5)$$

The same is true of the excess demand functions:

$$z_0(R_t) = x_0(R_t) - w \text{ and } z_1(R_t) = x_1(R_t) - w. \qquad (6)$$

Using the excess demand functions we can write the budget equality as

$$R_t z_0(R_t) + z_1(R_t) = 0. \qquad (7)$$

Let us now consider the "autarkic" consumption allocation, $x_t^t = w$ and $x_{t+1}^t = (1-w)$, and define $\overline{R}$ by

$$\frac{\left.\frac{fu}{fx_t^t}\right|_{(w,1-w)}}{\left.\frac{fu}{fx_{t+1}^t}\right|_{(w,1-w)}} = \overline{R}. \qquad (8)$$

It is immediate to see that the autarkic consumption is utility maximizing for $R = \overline{R}$. Further, by the definition of $\overline{R}$ we have that

$$z_0\left(\overline{R}\right) = z_1\left(\overline{R}\right) = 0. \qquad (9)$$

Notice now that the autarkic consumption belongs to the budget set for all *R*. Therefore, since for all $R\ ?\ \overline{R}$ the utility maximizing choice is given by $\left(z_0(R), z_1(R)\right)$ rather than by the autarkic allocation, it must be that

$$u\left(x_0(R), x_1(R)\right)\ ?\ u\left(w, 1-w\right) \text{ for all } R. \qquad (10)$$

By a standard revealed preference argument, we can deduce that the choices $\left(z_0(R), z_1(R)\right)$ must violate the budget constraint under $\overline{R}$. Therefore, we have that

$$\overline{R}z_0\left(R\right) + z_1(R) > 0 \text{ for all } R\ ?\ \overline{R}. \qquad (11)$$

Subtracting from the budget constraint (7) evaluated at R, we have

$$\left(\overline{R} - R\right)z_0\left(R\right) > 0 \text{ for all } R\ ?\ \overline{R}. \qquad (12)$$

Consumers have positive net demands when young (i.e. disave) when $R \leq \overline{R}$ and have negative net demands (i.e. save) when $R\ ?\ \overline{R}$. Further, it is easy to see that $z_0\left(R\right)$ is strictly decreasing in the neighbourhood of $\overline{R}$.

Let us now examine the existence and properties of a competitive equilibrium, i.e. a sequence of real interest rates *R* that clear the markets at every date *t*. Consider a particular period *t+1*. The relevant net demands are the ones expressed by the two consumers alive. Hence, the market clearing condition is

$$z_0\left(R_{t+1}\right) + z_1\left(R_t\right) = 0. \qquad (13)$$

Using the budget constraint (7) we can rewrite (13) as

$$z_0\left(R_{t+1}\right) - R_t z_0\left(R_t\right) = 0. \qquad (14)$$

Expression (14) is a difference equation characterizing the equilibrium sequence(s) *R*. A stationary solution R solves

$$\left(1 - R\right)z_0\left(R\right) = 0. \qquad (15)$$

Clearly, there are two stationary solutions: $R = \overline{R}$ and $R = R^* = 1$. The first solution corresponds to the autarkic consumption allocation. The second one, by inequality (12), will entail saving by the young $(z_0(1) \leq 0)$ if and only if $\overline{R} \leq 1$.

Let us now examine the local stability of the two steady-state solutions. To do so we just need to know the local properties of one of the two solutions. We chose to examine $\overline{R}$. Let us rewrite the equilibrium condition (14) as
$z_0(R_{t+1}) - z_0(R_t) = (R_t - 1)z_0(R_t)$. (16)

Let us suppose first that $\overline{R}$? 1. Consider any R_t the neighbourhood around $\overline{R}$ such that $\overline{R}$? 1. Let now $R_t > \overline{R}$, so that by (12) $z_0(R_t) < 0$. In view of (16) it follows that $z_0(R_t) > z_0(R_{t+1})$. Because of the strict decreasingness of z_0 around $\overline{R}$, we have that $R_t < R_{t+1}$. Let $R_t < \overline{R}$ instead. By the same arguments we deduce that $R_t > R_{t+1}$.

It is immediate that if we had $\overline{R} < 1$, by following the same steps as before, $R_t > R_{t+1}$ when $R_t > \overline{R}$ and $R_t < R_{t+1}$ when $R_t < \overline{R}$.

We thus have the following proposition:

Proposition 1: In the canonical OLG model there are twocompetitive equilibrium solutions. The barter solution is locally stable when $\overline{R} < 1$ $(z_0(1) \leq 0)$ and unstable if $\overline{R}$? 1 $(z_0(1) > 0)$.

Which equilibrium will prevail critically depends on the initial condition R_1. Suppose that $R_1 < \overline{R}$ so that $z_0(R_1) > 0$. Young consumers would have a strictly positive excess demand. If this were to be an equilibrium, this should be matched with an equivalent excess supply by the contemporary old. However, this cannot happen since this would be the first trade between this two consumers and hence the old could not carry over debts from previous contracts. By a symmetric argument, it cannot be that the young surrender part of their endowments to old consumers (with $R_1 > \overline{R}$), whom will not meet ever again. It follows that the only candidate equilibrium interest factor is $R_1 = \overline{R}$. The economy is tied to the competitive equilibrium yielding the worst utility possible (for consumers *t=1,2 …*) over all competitive consumption paths.

Is the competitive equilibrium Pareto efficient? Clearly not when $\overline{R} < 1$. To see this consider a deviation from the autarkic allocation consisting in transferring $\varepsilon > 0$ from the young to the old, starting in period *t=1*.This is a feasible reallocation. The old generation at *t=1* is strictly better-off. Further, for ε small enough, the new consumption allocation will be strictly preferred to autarky by any subsequent consumer. Hence, the competitive autarkic allocation cannot be Pareto efficient.

Proposition 2: In the canonical OLG model there is a unique competitive equilibrium: the one corresponding to the autarkic consumption allocation. Further, this competitive equilibrium fails to be Pareto efficient whenever it is locally stable.

Let us now consider the stationary consumption path associated to the interest factor $R^* = 1$. Suppose that $\overline{R} < 1$. At the equilibrium prices R^* excess supply by the young equals excess demand by the old. What prevents this allocation from being an equilibrium is that these trades cannot be carried out because the old generation has nothing worthy to the young generation to give in exchange.

It was Samuelson the first to conceive that this was precisely the role played by fiat money. Let the government issue M units of intrinsically worthless fiat money and give them to the old living in period $t=1$. Let the real interest factor be $R^* = 1$. At these prices, the young generation 1 will have an excess supply of the good $z_0(1)$.

Let q_1 be the amount of the good that can be purchased with 1 unit of money. Bearing in mind that $p_1=1$, let q_1^* be such that $q_1^* M = -z_0(1)$. It is then plain that $R^* = 1, q_1^* = -\frac{z_{0(1)}}{M}$, and M are a <u>monetary equilibrium</u> for the canonical OLG economy. Further, as we shall see in the next section, this consumption allocation is Pareto optimal. Notice that intrinsically worthless fiat money attains a strictly positive price on the basis of its role as a link between present and future markets.

Proposition 3: Whenever the autarkic competitive equilibrium fails to be Pareto efficient (alternatively, is locally stable), there exists a steady state competitive equilibrium supported with fiat money. The equilibrium price of money is strictly positive.

The steady state equilibrium with $R^* = 1$ is not the only monetary equilibrium possible. We have seen in Proposition 1 that, when $\overline{R} < 1$, the autarkic equilibrium is locally stable (and hence the steady state monetary equilibrium unstable). Therefore, if the initial condition at $t=1$ is not satisfied, the economy will not converge towards $R^* = 1$.

Let us consider q_1 ? q_1^*. The excess demand by the old who have benefited from the creation of fiat money will be $z_1^2 = q_1 M$. From the market clearing condition at $t=1$, we find that for each q_1 the equilibrium R must satisfy

$$z_0(R_1) = -q_1 M \ . \qquad (17)$$

It cannot be that q_1 is such that $R_1>1$ because condition (14) would require that the real interest factor grows unboundedly. This is clearly incompatible with feasible allocations. Let us consider instead q_1 such that $\overline{R} < R_1 < R^*$. It is plain that the sequence of interest factors will be decreasing and eventually converge towards $\overline{R}$. Notice that, since $\overline{R} \le R_t < 1$ along the sequence, $p_{t+1}>p_t$. Furthermore, by combining individual budget equalities with market clearing conditions, it can be readily verified that the equilibrium q will satisfy $q_t=q_1$, t=2,3,….. Hence the real value of money will be decreasing and converge to zero. In the long run, the consumption allocation will converge to the autarkic one.

Proposition 4: Whenever a steady state monetary equilibrium exists, there also exists a continuum of monetary equilibria. Along all these equilibrium paths, the real value of fiat money monotonically decreases and converges to zero.

Notice that any pair (q_1,R_1) satisfying (17) and such that $\overline{R} < R_1 \leq 1$ is a monetary equilibrium. We shall therefore have a continuum of monetary equilibria. The indeterminacy of equilibria is not a property specific to monetary economies. We also have indeterminacy in barter economies as soon as we depart from the canonical OLG model. It suffices to assume, for instance, that individuals live for three periods.

The existence of a continuum of equilibria is particularly problematic in OLG models. In order to solve the budget constrained maximization of utility, each consumer has to have a definite conjecture on the prices that will be ruling in the next period, as well as the current prices. But, will the conjectured prices for *t+1* become the actual equilibrium prices when the spot markets open at *t+1*? In our description of plans and exchange we have not distinguished between conjectured and realized prices. We have implicitly assumed that individuals were able to perfectly anticipate future prices, i.e. their expectations were "rational". In other environments it might be plausible that individuals could have attained a sufficient understanding of the background "true" model, enabling them to exactly predict future prices. However, in an OLG environment, even if individuals had a perfect knowledge of the "true" model, they would not be able to predict which equilibrium will occur because they are indeterminate.

Since Shell, Cass and Azariadis, there is a branch in the literature exploring the view that beliefs are an integral part of the definition of equilibrium. This allows for self-fulfilling prophecies tying future prices to the realitzation of events that are completely straneous to the economic problem under study. This type of beliefs have come to be known as "sun-spot" beliefs. This approach is appealing because it formalizes the Keynesian notion that economic cycles were caused by the "animal spirits" of investors. However, we shall not pursue this line here for two reasons. One is that this approach produces a mere "apparent" determinacy. There does not seem to be a theory of which straneous events are useable to anchor beliefs and which are not. How can one properly define the set of straeneous events? The second reason to leave "sun-spots" aside is that is that they are not specific to OLG models. We can have "sun-spot" equilibria in finite two-period economies with incomplete markets.

Our last point concerns the plausibility of the assumption that consumers behave competitively. We have already mentioned that in the standard GE the set of competitive equilibria belong to the core of the economy. This result is rightly interpreted as giving full coherence to the model. It does not pay to individuals to deviate from competitive behaviour by looking to alternative arrangements involving a subset of consumers only. Is this property inherited by the OLG models?

We know that the set of core allocations is a subset of the Pareto optimal allocations. It follows that inefficient autarkic competitive equilibria will fail to belong to the core of the economy. We have seen that in this case, there exist Pareto efficient allocations that can be achieved as monetary equilibria. Unfortunately, no allocation involving intergenerational transfers can belong to the core of the economy. The core, either contains the autarkic allocation only – when it is Pareto efficient-, or is empty.

To see this we start by observing that any allocation entailing transfers from the old to the young at some period *t+1* will be blocked by the old generation, who will refuse to carry out such transfer. Hence, any candidate allocation other than the autarkic one must involve non-negative transfers from the young to the old. Yet, consider the first generation performing a strict positive transfer when young. They will be strictly better off by excluding from the coalitions the previous generation and thus not carrying out the transfer. We conclude that the autarkic allocation is the only one that can belong to the core of the economy.

Proposition 5: The core of the canonical OLG model is either empty or it contains the autarkic allocation only.

It can be argued that the core is not a notion particularly appropriate for infinite horizon OLG economies. We shall discuss this issue in a later section. However, the previous result makes it plain that the assumption that in an OLG model individuals behave competitively is not on firm grounds. Alternatively, Proposition 5 indicates that OLG economies pose a problem that it is quite specific to them. This is the trustworthiness of trades involving the acquiescence of still unborn generations. Will future generations take on their shoulders the intergenerational transfers embodied in our current unfunded, pay-as-you-go pension system? Can we thus trust the current social contract implicit in the present system of taxes and (expected) future transfers?

To close this section let us summarize the main distinct features of the OLG model *vis-à-vis* the GE model. For the canonical OLG model we have obtained that competitive equilibria may fail to be Pareto efficient and hence may not belong to the core of the economy. Further, as soon as we depart from this canonical model competitive equilibria will typically be indeterminate. Whenever the competitive equilibrium is inefficient we shall also have distinct equilibria in which intrinsically worthless fiat money will attain a strictly positive price. However, the intergenerational trades involved will not be trustworhtly, in the sense that they will not belong to the core of the economy either.

All these features make the behaviour of the OLG model substantially different from the classical GE model. This differential behaviour constitutes a sort of puzzle. The canonical OLG model is a pure exchange economy with consumers endowed with well-behaved preferences. There are no externalities or non-convexities. The only difference appears to be that there is a countable infinity of individuals and of (dated) commodities, together with the overlapping structure of the population of consumers.

Some authors have placed the emphasis on the "large square" nature of OLG economies. Others have pointed to the incompleteness of markets. Finally, it has also been argued that the key feature is that individuals can permanently borrow from future generations. This leads to a lack of market clearing at infinity. We shall present here the complementary view that OLG economies can be understood as economies with a finite number of infinitely lived agents that are subject to periodical borrowing constraints.

We shall illustrate the point with an example. But it will be plain that the argument can be extended to any OLG economy. Let us start by considering an economy with one perishable good per period and two infinitely lived consumers. Suppose that their (identical) preferences can be represented by

$$\sum_{t=1}^{\infty} \delta^t \ln c_t \ .$$

Assume further that the stationary aggregate endowment of one unit is distributed in proportions *(w,1-w)* and *(1-w,w)* in alternating periods. It can be readily verified that there is a unique competitive equilibrium with prices $p_t=\delta^t$ and stationary consumption allocations

$$c^0 = \frac{\delta + (1-\delta)w}{1+\delta} \qquad c^e = \frac{1-(1-\delta)w}{1+\delta}.$$

where c^0 corresponds to the consumer with endowment w at $t=1$ (and c^e to the one with $(1-w)$). It is straightforward that this equilibrium is Pareto optimal. Notice that both consumption allocations tend to _ as $\delta \to 1$.

Consider now the equilibrium we would obtain if the *o* consumer is required to exactly balance her budget in every odd period, while consumer *e* has to balance the budget in every even period. For each consumer, the demand functions will now be obtained from a sequence of two-period constrained maximization problems. The resulting equilibrium conditions are <u>identical</u> to the ones we obtained when assuming that consumers "died" every two periods and that a new consumer was "born" in each period. The equilibrium prices and consumption allocation will exactly correspond to the ones we would obtain from the OLG analog. It is immediate to compute that the consumption allocations will be identical to their endowments, with equilibrium prices $p_t = \left(\delta \frac{w}{1-w}\right)^t$. Whenever, $w > \frac{1}{1+\delta}$ the resulting equilibrium consumption allocation will be inefficient.

We could as well have assumed three infinitely lived consumers with appropriate endowment sequences. For such an economy the equilibrium would still be unique and efficient. By imposing a similar structure of recurrent budget balancing constraints we would have the exact analogue of an OLG model with three-period lived consumers. We know that these economies have a continuum of equilibria.

It is plain that our example illustrates a general method for modelling analogs of OLG economies as infinitely lived consumer economies in which individuals are subject to recurrent balancing constraints. That consumers are born and die after a finite time is a very natural justification for the assumption that (lineage) budgets have to balance recurrently. The study of the general case in which different time patterns coexist and budgets need not to balance exactly is still an open question.

3. **Existence and efficiency of competitive equilibria**

We shall assume that there are l perishable commodities per period. Let $w_t \in \mathbb{R}^l_{++}$ be the aggregate endowment. We shall assume that this sequence is uniformly bounded from above by some finite $W \in \mathbb{R}^l_{++}$ and from below by $w>0$, $w \in \mathbb{R}^l_{++}$. We shall denote by w^t_{t+s}, $s=0,1$, the endowments of generation t in the two periods it is alive.

We shall denote by $x^t_{t+s} \in \mathbb{R}^l_{+}$, $s=0,1$, the consumption allocation in the two periods corresponding to individual t. We assume that individual preferences are intertemporally separable, i.e. $u^t(x^t) = f^t(x^t_t) + g^t(x^t_{t+1})$. As for the old generation at $t=1$, $u^0(x^0) = g^0(x^0)$. We shall assume that f and g satisfy: (i) strict increasingness; (ii) strict quasi-concavity (iii) twice differentiability and (iv) uniform boundedness of the gradient of f and g.

Assumption (iv) needs a more careful statement. We assume that for every ε>0, there exists $\lambda(\varepsilon)$ and $\overline{\lambda}$ such that

$$\lambda(\varepsilon) \le \min_t \left\| \frac{ff^t}{fx_t^t} \right\|, \left\| \frac{fg^t}{fx_t^{t+1}} \right\| ?? < \max_t \left\| \frac{ff^t}{fx_t^t} \right\|, \left\| \frac{fg^t}{fx_t^{t+1}} \right\| ?? \le \overline{\lambda}. \qquad (18)$$

For all vectors x_t^t, x_{t+1}^t such that $\varepsilon \le x_{t+s}^t \le N$, for all *t* and *s=0,1*.

Let $p_t \lfloor \Sigma_+^l$ be the price vector at *t* and *p* be the corresponding sequence $\langle p_t \rangle_{t=1}^{\times}$.

Let us now examine the consumer maximization problem. In the previous section we have argued that, because of the overlapping structure of a population made of individuals with finite lifetime, futures markets cannot exist. This is what precipitates the result that in the canonical OLG model the unique competitive equilibrium is the autarkic allocation. Let us momentarily take this observation on board and model individual choice as subject to a separate constraint for each period. We shall refer to these economies as "sequentially constrained" OLG economies. Accordingly, for given *p*, consumer demands are the solution to

max $f^t\left(x_t^t\right) + g^t\left(x_{t+1}^t\right)$

subject to

$p_t x_t^t \le p_t w_t^t$ and

$p_{t+1} x_{t+1}^t \le p_{t+1} w_{t+1}^t.$ (19)

Because of the additive separability of preferences, this is simply equivalent to solving two separate problems:

max $f^t\left(x_t^t\right)$ subject to $p_t x_t^t \le p_t w_t^t$ and

max $g^t\left(x_{t+1}^t\right)$ subject to $p_{t+1} x_{t+1}^t \le p_{t+1} w_{t+1}^t.$ (20)

The corresponding first order conditions are

$$\frac{ff^t}{fx_{t,i}^t} = \lambda_t^t p_{t,i} \quad i=1,\ldots.l. \qquad (21)$$

$p_t x_t^t = p_{t+1} w_t^t$ and

$$\frac{fp^t}{fx_{t+1,i}^t} = \lambda_{t+1}^t p_{t+1,i} \text{ , } i=1,\ldots,l \text{ .} \qquad (22)$$

$p_{t+1} x_{t+1}^t = p_{t+1} w_{t+1}^t.$

Clearly, both sets of solutions are homogeneous of degree zero in prices. Therefore, we shall normalize prices and restrict $\tilde{p}_t$ to belong to the unit sphere, i.e. $\|\tilde{p}_t\| = 1$, i=1,2… Notice that for normalized incomes, we shall have the Lagrange multipliers $\ddot{\tilde{\lambda}}_t^t$ and $\ddot{\tilde{\lambda}}_{t+1}^t$ satisfying

$$\tilde{\lambda}_t^t = \left\| \frac{ff^t}{fx_t^t} \right\| \text{ and } \tilde{\lambda}_{t+1}^t = \left\| \frac{fg^t}{fx_{t+1}^t} \right\|. \qquad (23)$$

Therefore, the Lagrange multipliers will be equal to the gradient of the utility function evaluated at the optimizing consumption bundle, that is, the norm of the first derivatives of the utility function. These Lagrange multipliers should be interpreted as the marginal utility of real income to consumer *t* in each period he is alive.

A competitive equilibrium for a sequentially constrained OLG economy is a sequence of consumption allocations $\langle \hat{x}_t \rangle_{t=1}^{\times}$ and (normalized) prices $\langle \tilde{p}_t \rangle_{t=1}^{\times}$ such that $\hat{x}_{t+s}^t$ is the solution to the consumer maximization problem given $\tilde{p}_{t+s}$ for *t=1,2…* and *s=0,1*, and

$$\hat{x}_t^t + \hat{x}_t^{t-1} = w_t^t + w_t^{t-1} \quad , t=1,2,\ldots \qquad (24)$$

We start by considering the existence of intra-period equilibria. This is a pure GE model. Under the present assumptions, it is a well-known result that equilibria exist and are locally unique in every period. Then, an equilibrium for the full sequentially constrained OLG economy consists of a sequence of one-shot equilibria for the periods *t=1,2…*. Notice that, if in a subsequence we have multiple one-shot equilibria, the full OLG economy will have a countable infinity of equilibria. The sequence of equilibrium prices $\tilde{p}$ will be a sequence of *l*-dimensional price vectors, each of which will belong to the unit sphere.

It is now straightforward to construct the price sequence *p* corresponding to the unconstrained competitive equilibrium. Our task is to find a sequence of intertemporal interest factors $\hat{R}_t$ such that the now unconstrained consumers will not be willing to shift wealth across periods.

Consider an arbitrary interest factor R_t. Recall that $\tilde{\lambda}_t^t$ and $\tilde{\lambda}_{t+1}^t$ are the marginal utilities of real income at *t* and *t+1*, evaluated at the equilibrium consumption allocation. Let us now consider the change in utility produced by the shifting of one unit of income between periods *t* and *t+1*: $-\tilde{\lambda}_t^t + R_t \tilde{\lambda}_{t+1}^t$. The equilibrium interest rate R_t clearly must satisfy

$$\hat{R}_t = \frac{\tilde{\lambda}_t^t}{\tilde{\lambda}_{t+1}^t} = \frac{\left\| \frac{ff^t(\hat{x}_t^t)}{fx_t^t} \right\|}{\left\| \frac{fg^t(\hat{x}_t^t)}{fx_{t+1}^t} \right\|}. \qquad (25)$$

Therefore, the (unconstrained) competitive equilibrium will be the same consumption allocation $\langle \hat{x}_t \rangle_{t=1}^{\times}$ and the sequence of prices $\langle p_t \rangle_{t=1}^{\times}$ given by

$$p_t = \tilde{p}_t \prod_{\tau=1}^{t-1} \frac{\tilde{\lambda}_{\tau+1}^{\tau}}{\tilde{\lambda}_{\tau}^{\tau}}, \; t=1,2,3\ldots \quad (26)$$

We now have the following proposition.

Proposition 6: Let individuals live for two periods and let there be *l* perishable commodities per period. Let endowments be uniformly bounded above and below away from zero. Let preferences be intertemporally additively separable and satisfy conditions (i), (ii), (iii) and (iv). Then, a competitive equilibrium exists. Further, except if preferences are restricted to give a unique one-shot constrained competitive equilibrium in all but a finite number of periods, we shall have a countable infinity of distinct equilibria.

Notice that all (unconstrained) competitive equilibria share one common feature. This is that because we have let consumers believe that future markets exist, the intertemporal equilibrium prices are such that no individual wishes to perform any net trade in these markets. Therefore, in spite of the fact that the competitive conjecture made by the consumers is not proved to be inappropriate in equilibrium, we must conceal that it rests on a very thin ground. A potentially fuitful implication of this observation is that competitive behaviour might not be such a reasonable assumption in a scenario in which intergenerational trades are involved. We shall come back to this point in a later section.

Let us now turn to the efficiency properties of competitive equilibria. Clearly, competitive allocations are efficient in the sense that they have supporting prices. In GE economies this property is equivalent to Pareto optimality. This is the Second Fundamental Theorem of Welfare Economics. This equivalence, however, is no longer true in OLG economies. Exploiting the existence of a countable infinity of consumers, it might be feasible to obtain a Pareto improvement by an appropriate sequence of income transfers from consumer *t+1* to the preceding consumer t. Since there is no last individual, the critical issue is to verify whether such a sequence of transfers can actually be carried out.

A feasible consumption allocation sequence $\langle x_t \rangle_{t=1}^{\infty}$ is said to be <u>Pareto Efficient</u> if it cannot be improved upon by any other feasible consumption allocation. It is said to be <u>Weakly Pareto Efficien</u>t if it cannot be improved upon by any other feasible consumption sequence involving reallocations over a finite number of periods only.

From the previous discussion it is easy to obtain the following result.

Proposition 7: Let individuals live for two periods and have inter-temporaly separable preferences. Let the sequence of endowments of the perishable commodities be uniformly bounded from above and below away from zero. Then, a feasible consumption allocation $\langle x_t \rangle_{t=1}^{\infty}$ is <u>Weakly Pareto Optimal</u> if and only if it is a competitive equilibrium relative to the endowment allocation $\langle w \rangle_{t=1}^{\infty}$ with $w_t = x_t$, *t=1,2,3…*

Let us now identify the Weakly Pareto Optimal allocations that are Pareto optimal. To this effect, let us consider a Weakly Pareto Optimal allocation $\left\langle x_t \right\rangle_{t=1}^{\times}$ and compute the sequence of transfers needed for a Pareto improvement.

Suppose that we transfer δ_1 from consumer $t=1$ to consumer $t=0$ in period $t=1$. Since utility is strictly increasing in real income, individual $t=0$ will be made strictly better-off. The transfer δ_1 made by consumer 1 will have to be compensated for in the next period with a received transfer of size δ_2 so as to keep his utility constant. Taking a linear approximation $\overline{\delta_2}$ to the true value of δ_2 we find that $\delta_1 \tilde{\lambda}_1^1 = \delta_2 \tilde{\lambda}_2^1$. By the same argument we obtain that

$$\overline{\delta_3} = \overline{\delta_2}\,\frac{\tilde{\lambda}_2^2}{\tilde{\lambda}_3^2} = \delta_1\,\frac{\tilde{\lambda}_1^1}{\tilde{\lambda}_2^1}\,\frac{\tilde{\lambda}_2^2}{\tilde{\lambda}_3^2}$$

It is easy to obtain for $\overline{\delta_t}$ that

$$\overline{\delta_t} = \delta_1 \subseteq_{\tau=1}^{t-1} \frac{\tilde{\lambda}_\tau^\tau}{\tilde{\lambda}_{\tau+1}^\tau}. \qquad (27)$$

Taking into account that $\tilde{p}_t$ belongs to the unit sphere for all t and using (26), we have that

$$\subseteq_{\tau=1}^{t-1} \frac{\tilde{\lambda}_\tau^\tau}{\tilde{\lambda}_{\tau+1}^\tau} = \frac{1}{\left\| p_t \right\|}. \qquad (28)$$

It follows that

$$\overline{\delta_t} = \delta_1 \frac{1}{\left\| p_t \right\|} \quad (29)$$

Since $\overline{\delta_t}$ is a lower estimate of the true value of δ_t, it suffices that the norm of prices goes to zero as t goes to infinity to make the sequence of transfers unbounded. Clearly such a sequence can not be feasible in an economy with uniformly bounded resources.

Proposition 8: The condition

$$\lim_{t \blacklozenge 0} \inf \left\| p_t \right\| = 0. \qquad (30)$$

is sufficient for the Pareto efficiency of the corresponding consumption allocation sequence.

The condition in Proposition 8 is sufficient –but not necessary- for Pareto optimality because our linear approximation to the sequence of compensating transfers has disregarded the terms of second order. Note this error has compounded over an infinite sequence of approximations and it might have become significant. The following proposition gives a necessary and sufficient condition for Pareto efficiency.

Proposition 9: The feasible consumption allocation sequence is Pareto optimal if and only if

$$\sum_{t=1}^{\infty} \frac{1}{\|p_t\|} = +\infty \ . \qquad (31)$$

Consider the condition given in Proposition 8. In view of (28), a sufficient condition for Pareto optimality is that $\tilde{\lambda}_\tau^\tau > \tilde{\lambda}_{\tau+1}^\tau$. In other words, we require that the compensation tomorrow for today's transfers be more than proportional.

4. **Competitive equilibria with fiat money**

In the previous section we have seen that, even when we assume the existence of a complete array of markets, universally beneficial reallocations of resources might not be accessible to individuals who behave competitively. Due to the finite lifetime of consumers, competitive equilibria are compatible with the use of spot markets only. This is so because, when generations partially overlap, intertemporal trades necessarily involve intergenerational trades. The role of money can thus be seen as a way of overcoming the impossibility of contracting with future, still unborn individuals.

The existence of money permits individuals to transfer purchasing power from present to future periods in their lifetime. In a monetary economy individuals are no longer constrained to consume no more than their current resources in every period. With two-period lived individuals, consumers can when young purchase money as well as the real commodities currently available. This money will be sold in the second period for additional consumption goods.

Let m_t^t be the demand for money by consumer *t* when young. The demands for commodities and money are given by the solution to

$$\max f^t\left(x_t^t\right) + g^t\left(x_{t+1}^t\right), \text{ subject to}$$
$$p_t x_t^t + m_t^t \leq p_t w_t^t \text{ and}$$
$$p_{t+1} x_{t+1}^t \leq p_{t+1} w_{t+1}^t + m_t^t \qquad (32)$$

By adding up these two constraints we can alternatively rewrite the maximization problem as

$$\max f^t\left(x_t^t\right) + g^t\left(x_{t+1}^t\right) \text{ subject to}$$
$$p_t x_t^t + p_{t+1} x_{t+t}^t \leq p_t w_t^t + p_t w_{t+1}^t \text{ and } m_t^t \ ? \ 0 \, . \qquad (33)$$

The demand for money is either zero or is identical to the aggregate value of the excess supply in the first period, when this is positive.

At $t=1$ the then old generation will be carrying the existing stock of money, m_0^0, and their demands will satisfy

$$p_1 x_1^0 = p_1 w_1^0 + m_0^0. \qquad (34)$$

Hence, $m_0^0 \ldots m$ will be the inelastic supply of money at $t=1$.

A competitive monetary equilibrium is a sequence of consumption allocations $\langle \hat{x}_t \rangle_{t=1}^{\infty}$, money demands $\langle m_t \rangle_{t=1}^{\infty}$, and prices $\langle p_t \rangle_{t=1}^{\infty}$ such that

$$\hat{x}_t^t + \hat{x}_t^{t-1} = w_t^t + w_t^{t-1} \text{ and } m_t=m,\ t=1,2,3 \ldots \qquad (35)$$

In order to better understand the role of money and the conditions for the existence of monetary equilibria we shall take the convention that consumers are subject to two independent budget constraints. Let us consider a specific competitive equilibrium allocation $\langle \hat{x}_t \rangle_{t=1}^{\infty}$ with prices $\langle \hat{p}_t \rangle_{t=1}^{\infty}$. When fiat money is available consumers can shift future nominal income into present nominal income at an implicit interest factor $R^m=1$. The key question is therefore whether there will be a positive demand for an asset with this return. Let us start by considering that consumers take m as given. Their nominal maximization problem is still subject to two independent constraints that have now been shifted by m. For prices $\hat{p}$ and $m=0$ the corresponding demands for consumption goods will coincide with the competitive equilibrium ones. The point is whether individual utility will be increased by letting $m>0$. The effect of dm on individual utility clearly is

$$du^t = -\tilde{\lambda}_t^t dm + \tilde{\lambda}_{t+1}^t dm = \left(\tilde{\lambda}_{t+1}^t - \tilde{\lambda}_t^t\right) dm. \qquad (36)$$

Therefore $du^t>0$ with $dm>0$ if and only if $\frac{\tilde{\lambda}_{t+1}^t}{\tilde{\lambda}_{t+1}^t} = \hat{R}_t < 1$. Notice now that in order to have a monetary equilibrium we need that at the (barter) competitive equilibrium prices every consumer $t=1,2,3\ldots$ demands a strictly positive amount of money. Let us denote by $\frac{\overline{\lambda}_{t+1}^t}{\overline{\lambda}_{t+1}^t}$ the sequence of intertemporal rates of substitution that are maximal at every t over the possibly multiple barter equilibria. We then have the following result.

Proposition 10: A necessary condition for the existence of a monetary equilibrium is that

$$\limsup_{t \to \infty} \frac{\overline{\lambda}_t^t}{\overline{\lambda}_{t+1}^t} < 1.$$

Therefore , a necessary condition for the existence of a monetary equilibrium is that the sequence of (barter) equilibrium $\frac{\tilde{\lambda}_{t+1}^{t}}{\tilde{\lambda}_{t+1}^{t}}$ is uniformly bounded above by some $\overline{R} < 1$.

Let $\langle p_t \rangle_{t=1}^{\times}$ be any (barter) equilibrium price sequence. Using (28) and Proposition 10 we have that

$$\frac{1}{\|p_t\|} \leq \underset{\tau=1}{\overset{t-1}{\subseteq}} \overline{R} = \overline{R}^{\,t-1} . \qquad (38)$$

for all barter competitive equilibrium price sequences.

Adding up,

$$\underset{t=1}{\overset{\times}{}} \frac{1}{\|\hat{p}_t\|} = \frac{1}{\overline{R}} \underset{t=1}{\overset{\times}{}} \overline{R}^{\,t} = \frac{1}{1-\overline{R}} . \qquad (39)$$

Proposition 11: A necessary condition for the existence of a monetary equilibrium is that all the (barter) competitive equilibria fail to be Pareto efficient.

Let us now turn to the existence of monetary equilibria. In order to clearly understand the role of money and the conditions for the existence of a monetary equilibrium, we shall follow the same steps as in the existence of (barter) competitive equilibria. The argument there was that in fact individuals cannot transfer purchasing power across periods, but were subject to a sequence of budget constraints. Therefore, the sole point was to find a sequence of market clearing spot prices. The intertemporal profile of the price sequence was irrelevant. However, the notion of (unconstrained) competitive behaviour could still be savaged by setting an intertemporal profile in the price sequence such that at the chosen interest factors it is optimal to every consumer not shift income across periods. In a similar vein, we can conceive individuals as subject to a sequence of budget constraints, each limited by the existing stock of money. In a second stage, we find the interests factors for which the induced demands for money do coincide with the given supply, *m*.

We thus start by assuming that, taking *m* as given, individuals choose (x_t^t, x_{t+1}^t) that solve

max $f^t(x_t^t)$ subject to $p_t x_t^t \leq p_t w_t^t - m$, and

max $g^t(x_{t+1}^t)$ subject to $p_{t+1} x_{t+1}^t \leq p_{t+1} w_{t+1}^t + m.$ (40)

Normalizing the prices to the unit spheres, $\tilde{p}$, the first order conditions for a maximum are

$\frac{ff^t(x_t^t)}{fx_{t,i}^t} = \tilde{\lambda}_t^t \tilde{p}_{t,i}$, i=1,…,l, $\tilde{p}_t x_t^t = \tilde{p}_t w_t^t - \frac{m}{\|p_t\|}$ and

$$\frac{fg^t\left(x_{t+1}^t\right)}{fx_{t+1,i}^t} = \tilde{\lambda}_{t+1}^t \tilde{p}_{t+1,i}\,,\ i=1,\ldots l,\ \tilde{p}_{t+1}x_{t+1}^t = \tilde{p}_{t+1}w_{t+1}^t - \frac{m}{\left\|p_{t+1}\right\|}. \qquad (41)$$

The resulting demand functions will depend on the normalized price vector, $\tilde{p}$, and the "real" value of money $\frac{m}{\left\|p_t\right\|}$ at each period.

We shall choose $\left\|p_1\right\| = 1$. The price vector $\tilde{p}_1$ will be an equilibrium for the spot markets at *t=1* if

$$x_1^0\left(\tilde{p}_1, m\right) + x_1^1\left(\tilde{p}_1, m\right) = w_1^0 + w_1^1. \quad (42)$$

The role of money is equivalent to a reallocation of the endowments between the two generations. Therefore, when individuals take *m* as a given parameter, the economy at period *t=1* can be separated from the rest of the sequence and be taken as a static, one-period pure exchange economy. Under the current assumptions, it is a known result that an equilibrium $\tilde{p}_1$ (possibly finitely many) exists. The equilibrium price vector will depend on *m*, $\tilde{p}_1(m)$.

Keeping in mind that *m* is considered as exogenous, the demands for period *t=2* will result from

$$\max f^2\left(x_2^2\right) \text{ subject to } \tilde{p}_2 x_2^2 \le \tilde{p}_2 w_2^2 - \frac{m}{\left\|p_2\right\|} = \tilde{p}_2 w_2^2 - R_1 m \text{, and}$$

$$\max g^1\left(x_2^1\right) \text{ subject to } \tilde{p}_2 x_2^1 \le \tilde{p}_2 w_2^1 + \frac{m}{\left\|p_2\right\|} = \tilde{p}_2 w_2^1 + R_1 m \quad (43)$$

Again, for each given $R_1 m \le \left\|w_2^2\right\|$, there exist an equilibrium $\tilde{p}_2(R, m)$ for the spot markets at *t=2*.

In general, for any *t* and for each sequence $\left\langle R_\tau \right\rangle_{\tau=1}^{t-1}$ such that $\underset{\tau=1}{\overset{t-1}{\subseteq}} R_\tau m < \left\|w_t^t\right\|$, there exists an equilibrium $\tilde{p}_t\left(\underset{\tau=1}{\overset{t-1}{\subseteq}} R_\tau m\right)$ clearing the spot markets at *t*.

Notice that

$$\underset{\tau=1}{\overset{t-1}{\subseteq}} R_\tau m = \frac{m}{\left\|p_t\right\|} < \left\|w_t^t\right\|. \qquad (44)$$

implies that all potential intertemporal price sequences are uniformly bounded below away from zero. However, for given *M* and for all such sequences a <u>constrained</u> "monetary" equilibrium exists.

Now it remains to be shown that among such sequences there is at least one such that all agents would demand nominal money by an amount m, had they not treated m parametrically. This will be the case when for each generation the intertemporal marginal rate of substitution (evaluated at the corresponding consumption vector) is equal to the relevant interest factor.

Let us denote by $m^t = \subseteq_{\tau=1}^{t-1} R_\tau m$. Now define

$$\phi^t \left(R_t ?m^t\right) = R_t \tilde{\lambda}_{t+1}^t \left(\tilde{p}_{t+1}, \tilde{p}_{t+1} w_{t+1}^t + R_t m_t\right) - \tilde{\lambda}_t^t \left(\tilde{p}_t, \tilde{p}_t w_t^t - m_t\right). \qquad (45)$$

A sequence $\langle R_\tau \rangle_{t=1}^{\times}$ is an equilibrium if $\phi^t\left(R_t ?m^t\right) = 0$ $t=1,2,3\ldots$ In order to see that such an sequence exists, let us start by evaluating (45) at the barter equilibrium interest factor R_t^b. Consider first (45) at R_t^b and $m_t=0$. We have

$$\phi^t \left(R_t^b ?0\right) = R_t^b \tilde{\lambda}_{t+1}^t \left(\tilde{p}_{t+1}^b, \tilde{p}_{t+1}^b w_{t+1}^t + R_t m_t\right) - \tilde{\lambda}_t^t \left(\tilde{p}_t^b, \tilde{p}_t^b w_t^t\right) = 0 . \qquad (46)$$

For any $m_t>0$, and because of the decreasing marginal utility of income, we shall have that

$$\phi^t \left(R_t ?m_t\right) < 0$$

Consider now R_t^a solving

$$R_t^a \lambda_{t+1}^t \left(\tilde{p}_{t+1}, \tilde{p}_{t+1} ?w_{t+1}^t + m_t\right) = \tilde{\lambda}_t^t \left(\tilde{p}_t, \tilde{p}_t ?w_t^t - m_t\right). \qquad (47)$$

Such a R_t^a clearly exists. Further, $R_t^a \blacklozenge R_t^b$ as $m_t \blacklozenge 0$.

Hence, for $m_t>0$ small enough we shall have that

$$R_t^b < R_t^a < 1 . \qquad (48)$$

Observe now that, using (47), we have

$$\phi^t \left(R_t^a ?m^t\right) = R_t^a \tilde{\lambda}_{t+1}^t \left(\tilde{p}_{t+1}, \tilde{p}_{t+1} w_{t+1}^t + m_t - \left(1 - R_t^a\right) m_t\right) - \tilde{\lambda}_t^t \left(\tilde{p}_t, \tilde{p}_t w_t^t - m_t\right).$$

By (48) it follows that

$$\phi^t \left(R_t^a ?m_t\right) > 0$$

By the continuity of ϕ^t with respect to R_t it follows that for every m^t, there exists $\tilde{R}_t$ such that

$$\phi^t \left(\breve{R}_t ?m_t\right) = 0 \text{ and } R_t^b < \breve{R}_t < R_t^a < 1$$

Notice now that by using $\tilde{R}_t$ in every period we shall have

$$\tilde{m}_t = \prod_{\tau=1}^{t-1} \tilde{R}_t m < m \, . \qquad (49)$$

Therefore, for such sequence of interest factors there always exists $m>0$ small enough such that $R_t^a < 1$ for $t=1,2\ldots$ Further, the restriction (44) can be satisfied as well. Finally, notice that m is the value (at the equilibrium price for fiat money) of the money supply.

Thus we have the following Proposition.

Proposition 12: Let all the barter competitive equilibria be inefficient. Then, a monetary equilibrium always exists.

Observe that the monetary equilibrium we have shown to exist has interest factors uniformly strictly less than unity. It follows that this monetary equilibrium will fail to be Pareto efficient. Furthermore, the corresponding price sequence $\langle p_t \rangle_{t=1}^{\infty}$ grows unboundedly and therefore the real value of the money supply, $\frac{m}{\|p_t\|}$, tends to zero as $t \to \infty$. The equilibrium consumption allocations converge towards the barter equilibrium allocation.

Are there "regular" monetary equilibria with $\frac{m}{\|p_t\|}$ uniformly bounded below away from zero? It is immediate that if such regular equilibria exist the corresponding price sequence has the property that $\frac{1}{\|p_t\|}$ is uniformly bounded away from zero and hence the equilibrium allocations would be Pareto optimal. The existence of regular monetary equilibria is problematic at the present level of generality. However, they can be shown to exist for economies with stationary preferences and endowments.

The existence of monetary equilibria shows that the introduction once and for all of a worthless object can have real effects. Does this result justify monetary activism by governments? In other words, does the varying of the money supply have real effects? With a non-constant money stock we define m_t to be the aggregate money supply at period t. m_t-m_{t-1} is the subsidy (if positive) or tax (if negative) in nominal money on generation $t-1$ at period t. Assume now that preferences and endowments are stationary from generation to generation. Then, the following proposition can be demonstrated.

Proposition 13: Let the economy be stationary in preferences and endowments. Let all the barter competitive equilibria be inefficient. The, there exists an efficient regular monetary equilibrium for the monetary policy $\langle m_t \rangle_{t=1}^{\infty}$ if:

1. $\liminf_{t \blacklozenge \times} \frac{m_{t+1}}{m_t} > 0$

2. $\limsup_{t \blacklozenge \times} \frac{m_{t+1}}{m_t} < \frac{1}{R^b}$; and

3. $\overset{\times}{\underset{t=1}{}} \frac{1}{m_t} = +\times$

where R^b is the largest equilibrium interest factor for the barter economy.

Let us briefly discuss the structure of monetary equilibria. We know that regularity implies that $\frac{m_t}{\|p_t\|}$? $\varepsilon > 0$. Further, by feasibility the real value of money cannot exceed the resources, that is, $\frac{m_t}{\|p_t\|} < W$, where W is the uniform upper bound on available resources. Therefore, it follows that $\dfrac{\frac{m_{t+1}}{m_t}}{\frac{\|p_{t+1}\|}{\|p_t\|}}$ has to be uniformly bounded above and below. Let us consider the simple rule $\frac{m_{t+1}}{m_t} = \frac{\|p_{t+1}\|}{\|p_t\|}$. This clearly satisfies the boundary conditions. It simply says that the monetary policy consisting in keeping the real value of money constant generates a regular monetary equilibrium. Note, however, that this policy entails increasing the money supply whenever there is inflation. This implication is quite puzzling from the policy point of view, but is clearly consistent with the logic behind OLG models.

Indeed, the OLG model with rational expectations stands for being the only one that can endogenously explain the valuation of intrinsically worthless fiat money. However, in its present form, it appears to be missing fundamental ingredients of the determinants of the role of money in the economy.

5. **Intergenerational transfers and trust**

In section 2 we have mentioned the fact that in the canonical OLG model the core of the economy is either empty or it contains the endowment allocation (when it is efficient). Equilibrium competitive net trades might not be trustworthy since future generations many renege on obligations established before they were born. Indeed, in the canonical OLG model no intergenerational transfer is trustworthy.

In the present section we shall discuss two issues. The first one is the eventual extension of this negative result to more general specifications of the OLG model. The second issue concerns the appropriateness of the core as a suitable concept to capture the trustworthiness of agreements in an OLG environment.

We start by assuming one individual per generation with two-period lifetime. There are *l* non-storable commodities. Preferences are supposed to satisfy the assumptions introduced in section 3. The sequence of endowments allocations is uniformly bounded from above and from below away from zero.

Our first notion of trustworthiness (the core) is as follows. Consider a potential agreement involving the two consumers alive at *t=1* and departing from the barter equilibrium allocation. It clearly cannot make the young consumer better-off at the expense of the old consumer. One can conceive however that the young consumer decrease his utility today and benefit the old consumer in exchange for a compensation tomorrow. Can this intergenerational transfer be trustworthy? Individual *t=1* will have to verify whether such compensating transfer would be agreeable to individual *t=2*. This in turn requires verifying its acceptability to individual *t=3*, and so on. A very weak notion of trustworthiness is that a proposed consumption allocation is acceptable if there does not exist any subset of individuals and a consumption allocation feasible to them such that all are at least as well off and some strictly better off than in the proposed allocation. In other words, an individual will head a blocking deviation if he can find an alternative feasible allocation that is as acceptable as the original one to future generations, and strictly better for himself. This is but the core for an OLG economy.

More formally, the consumption allocation $\langle x_t \rangle_{t=1}^{\times}$ belongs to the core of the economy if there is no subset of individuals *S* and allocation $\langle \hat{x}_t \rangle_{t=1}^{\times}$ feasible for all $t \in S$ such that

$$f\left(\hat{x}_t^t\right) + g\left(\hat{x}_{t+1}^t\right) ?\ f\left(x_t^t\right) + g\left(x_{t+1}^t\right),\ \forall t \in S. \qquad (50)$$

Suppose that individual *t* decides to head a coalition blocking the allocation *x* with $\hat{x}$. Since individual *t* has excluded the previous generation from the new arrangement, it must be that $\hat{x}_t^t = w_t^t$ because of feasibility. He will head a coalition with the alternative allocation $\hat{x}$ only if.

$$f^t\left(w_t^t\right) + g^t\left(\hat{x}_{t+1}^t\right) > f^t\left(x_t^t\right) + g^t\left(x_{t+1}^t\right). \qquad (51)$$

Combining (50) and (51) for all generations t, one can readily obtain the following result.

Proposition 14: A feasible consumption allocation sequence belongs to the core of the economy if and only if it is Pareto-optimal and

$$f^t\left(x_t^t\right) ?\ f^t\left(w_t^t\right) \text{ and } g^{t-1}\left(x_t^t\right) ?\ g^{t-1}\left(w_{t-1}^t\right),\ t=1,2\ldots$$

From Proposition 14 it follows that whenever the barter equilibrium is Pareto optimal, it belongs to the core of the economy. But, this simply says that trades involving no intergenerational transfers are trustworthy.

Observe that this result also tells us that some transfers (backed with money) might be trustworthy as long as the size of the transfer is not "too large". Specifically, the condition in Proposition 14 says that the size (in utility) of the intergenerational transfers cannot exceed the size of the utility gains generated by the intra-period trades.

Trustworthiness becomes more problematic when generations are composed of more than one consumer. To see this, consider the following example of a stationary economy with one good with aggregate endowment of two units per period. There are two individuals per generation with identical preferences,

$$u(x^t) = f(x_t^t) + f(x_{t+1}^t).$$

Suppose that consumer *a* has shares α and $(1-\alpha)$ of the aggregate endowment when young and old, respectively, while *b* has shares $(1-\alpha)$ and α. Let α>_. Clearly, the consumption allocation $x_t^{t,a} = x_{t+1}^{t,a} = x_t^{t,b} = x_{t+1}^{t,b} = 1$ is a Pareto optimal, competitive equilibrium for this economy. Pareto optimality follows from the fact that the equilibrium prices satisfy $p_t=1$, for all *t*.

In this equilibrium, consumer *a* lends $(2\alpha-1)$ to consumer *b* when both are young and is repaid by *b* when both are old. Thus, this is an economy with <u>inside</u> money. As it turns out, this efficient, competitive allocation does not belong to the core. Indeed, consumer *a* of generation $t=1$ will be strictly better-off by initiating a coalition of the a-type consumers with the allocation $(2\alpha,1)$ for himself and $(1,1)$ for the subsequent coalition members.

Proposition 15: Let there be more than one agent per generation. Then, there are OLG economies with a non-empty core such the Pareto optimal barter competitive equilibrium does not belong to the core.

The potential untrustworthiness of efficient competitive trades is not restricted to the case of intergenerational transfers. We also have that futures contracts with other members of the same cohort might not be trustworthy. The intuitive reason is that substituting fiat money for inside money creates a seigniorage gain, that some consumers wish to capture. This problem is endemic when we let the number of individuals per generation become arbitrarily large.

Proposition 16: For large OLG economies a Pareto optimal competitive equilibrium belongs to the core if and only if it does not involve the use of either outside (fiat) or inside money in the long-run.

In the GE model, the fact that competitive equilibria belong to the core is interpreted as making logically coherent the hypothesis that individuals behave competitively. It is plain that in OLG economies either the assumption of competitive behaviour or the notion of trustworthiness (as captured by the core) or both, are not appropriate.

One of the lines followed in the literature has been to reformulate the notion of the core for OLG economies. The aim is to obtain the result that, as in the GE model, competitive equilibria belong to the set of “revised core” allocations. The attempts so far have not been quite satisfactory.

Let us now examine the case in which individuals act strategically –rather than competitively- with respect to future, unborn generations. We shall use the simple canonical OLG model of section 2. Preferences and endowments are stationary. There is one perishable good per period and one individual per generation. Suppose that the initial endowment allocation $(w,1\text{-}w)$ is not Pareto efficient. We know that there is a feasible transfer δ such that the resulting consumption allocation $(w\text{-}\delta,1\text{-}w+\delta)$ Pareto dominates the initial one and is Pareto optimal. Consider the case in which there are two strategies: perform and not to perform the transfer δ. It is immediate that not to perform the transfer is the unique equilibrium in dominant strategies. Indeed, irrespective from the strategy followed by the next generation, not performing the transfer is always a best response.

Proposition 17: For a canonical OLG model, let individual strategies consist in either performing or not to a given strictly positive transfer δ. Then not performing the transfer is the unique equilibrium in dominant strategies.

Once again we obtain the result that intergeneration transfers are not trustworthy. We can object to the previous specification of strategic behaviour that it overlooks the plausible time-dependence of strategies. The willingness to accept a given transfer to our parents may depend on how they behaved towards their own parents. Notice that this strategic linkage plays a critical role on two counts. First, and most obvious, individuals will more easily accept high transfers when they observe that their predecessors too agreed on high transfers. Secondly, there is the more subtle strategic argument that, by we giving a large transfer, we are making more likely that our children will in their turn accept a large transfer to us. The strategic analysis of intergenerational transfers is an important open problem.

6. Concluding remarks on future research

The OLG modeling of the economy seems particularly suited for a rich variety of problems that involve actions whose consequences extend into the future. Research has so far focused on the most standard cases of intergenerational transfers such as money, public debt and social security. There are many more areas in which the explicit modeling of the overlapping cohort structure of the population is relevant, but have received comparatively little attention. Examples abound. Exploitation of non-renewable or partially renewable resources, the generation of negative external effects on unborn generations (pollution and global warming), or of positive ones like investments in R&D benefiting future generations that have not participated in the cost are natural examples. But we could also include the intergenerational transmission of values, norms and habits that will condition the decisions of our descendants.

At a more theoretical level, the results we have reported can be interpreted as revealing fundamental shortcomings in the current modeling of intergenerational interactions. Indeed, it seems inappropriate to assume that individuals make plans as if they could trade with unborn agents at perfectly foreseen prices that are treated parametrically. The counterpart to this shortcoming is that the thus generated actions may not be trustworthy. This is not meant to imply that the particular notion of trust we have been using --the core-- is the most natural one in this context. In fact, the notion of how do we act vis-à-vis unborn generations cannot be dissociated from the appropriate specification of what will be found acceptable by these future generations when they come into play. This area of research appears to be essential for a deeper understanding of the economic interactions between generations.

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The OLG model initiates in Samuelson's (1958) seminal paper. Cass and Yaari (1966) and Gale (1973) provide a good and clear account of the OLG model. Cass-Okuno-Zilcha (1979) introduced the "reflected offer curve" as a useful device for the analysis of existence and stability of equilibria.

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2. Intertemporal efficiency

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3. Competitive Barter Equilibrium. Existence, efficiency and indeterminacy.

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Balasko-Shell (1981) examine properties of monetary equilibria for the case of many commodities. The first proof of existence of an efficient monetary equilibrium in a stationary OLG model with a constant stock of money is due to Benveniste and Cass (1986). The case of "active" monetary policies, in which the money supply may vary from period to period, has received little attention. Besides Balasko and Shell (1981) and (1986), there are the contributions of Burke (1987) and (1988) and Esteban, Mitra and Ray (1994). Burke proves the existence of efficient moneraty equilibria for monetary policies consisting in the introduction of money followed by a sequence of budget surpluses. Esteban, Mitra and Ray (1994) completely characterize the sequences of money supply for which an efficient monetary equilibrium exists.

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6. Production and Capital Accumulation

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